ETH Zürich FS 2015
D-MATH
Prof. Dr. Josef Teichmann

Coordinator
Ren Liu

Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 8

1. Recall that for a stopping time τ and a process $(M_t)_{t\geq 0}$ the stopped process is defined by $(M_t^\tau)_{t\geq 0} = (M_{\tau\wedge t})_{t\geq 0}$.
For $K > 0$, we consider the stopping time $\sigma_K := \inf\{t > 0 | \langle M\rangle_t > K\}$. Since $\langle M\rangle$ is continuous, we have that $\langle M\rangle_t \leq K$ for $t \leq \sigma_K$, and therefore

$$E\big[\langle M^{\sigma_K}\rangle_\infty\big] = E\big[\langle M\rangle_{\sigma_K}\big] \leq K.$$

Hence, Ex 7-2a) gives that $M^{\sigma_K} \in \mathcal{H}_0^{2,c}$. We can therefore apply Tchebycheff's and Doob's inequality (and use that the constant in Doob's inequality for fixed $p > 1$, denoted by C_p, is equal to $\big(\frac{p}{p-1}\big)^p$), obtaining that

$$\begin{aligned} P\big[(M^{\sigma_K})_t^* > C\big] &\leq \frac{E\big[((M^{\sigma_K})_t^*)^2\big]}{C^2} \\ &\leq \frac{4E\big[(M^{\sigma_K})_t^2\big]}{C^2} \\ &= \frac{4E\big[\langle M^{\sigma_K}\rangle_t\big]}{C^2} \\ &\leq \frac{4K}{C^2}. \end{aligned}$$

To obtain the claim, we observe that

$$\{M_t^{\sigma_K} \neq M_t\} \subseteq \{\sigma_K < t\} = \{\langle M\rangle_t > K\},$$

which finally implies that

$$\begin{aligned} P\big[M_t^* > C\big] &= P\big[M_t^* > C, \sigma_K \geq t\big] + P\big[M_t^* > C, \sigma_K < t\big] \\ &\leq \frac{4K}{C^2} + P\big[\langle M\rangle_t > K\big]. \end{aligned}$$

2. **a)** Without loss of generality, suppose $(X_t)_{t\geq 0}$ is a local martingale with $X_0 = 0$ and let B be a constant such that $|X_t| \leq B$ for all $t \geq 0$. Let $(\tau_k)_{k\in\mathbb{N}}$ be a

Bitte wenden!

localizing sequence for X, i.e. it is a non decreasing sequence of stopping times such that $(X_{t\wedge\tau_k})_{t\geq 0}$ is a martingale for any k and $\tau_k \nearrow +\infty$ a.s. Fix $s \leq t$, by the martingale property we have

$$E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = X_{s\wedge\tau_k} \text{ a.s.}$$

By dominated convergence theorem, which we can apply by the uniform boundedness of X, we get that

$$E\big[X_t \,\big|\, \mathcal{G}_s\big] = \lim_{k\to\infty} E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = \lim_{k\to\infty} X_{s\wedge\tau_k} = X_s \text{ a.s.}$$

b) Let $(\tau_k)_{k\in\mathbb{N}}$ be a localizing sequence for X. Then, applying the local martingale property, we have for any $0 \leq s \leq t \leq T$ that

$$X_{s\wedge\tau_k} = E[X_{t\wedge\tau_k}|\mathcal{G}_s] \quad \text{a.s.}$$

Since X is nonnegative, we can apply Fatou's lemma to get for any $0 \leq s \leq t \leq T$ that

$$X_s = \lim_{k\to\infty} X_{s\wedge\tau_k} = \liminf_{k\to\infty} E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] \geq E\big[\liminf_{k\to\infty} X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = E\big[X_t|\mathcal{G}_s\big] \quad \text{a.s.} \tag{1}$$

Moreover as X is nonnegative, we obtain by applying Fatou's Lemma that for any $t \in [0, T]$

$$E\big[X_t\big] = E\big[\liminf_{k\to\infty} X_{t\wedge\tau_k}\big] \leq \liminf_{k\to\infty} E\big[X_{t\wedge\tau_k}\big] = E\big[X_0\big] < \infty$$

and so $(X_t)_{t\geq 0}$ is a supermartingale.
Now, take the expectation on both sides in (1), we get

$$E[X_s] \geq E[X_t]$$

for all $0 \leq s \leq t \leq T$. In particular, using monotonicity of the expectation for a supermartingale, we have

$$E[X_0] \geq E[X_s] \geq E[X_t] \geq E[X_T] \quad \text{for all } 0 \leq s \leq t \leq T. \tag{2}$$

Using the assumption $E[X_T] = E[X_0]$, we see that the previous inequalities in (2) are all equalities. If the inequality in (1) was strict on a set of positive probability, we would have $E[X_s] > E[X_t]$, which gives a contradiction, and so the equality must hold with probability one. Thus, X is a martingale.

c) As M is a continuous local martingale, there exists by definition a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$, P-a.s. tending to infinity, such that for each n, the stopped process $M^{T_n} := (M_{T_n\wedge t})_{t\geq 0}$ is a continuous martingale. Consider the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\big\{t \geq 0 \,\big|\, |M_t| > n\big\}$$

Siehe nächstes Blatt!

As M_0 is bounded, we get that P-a.s. τ_n tends to infinity. Moreover, by (left-) continuity of M, for any n, the process $M^{\tau_n} := (M_{\tau_n \wedge t})_{t\geq 0}$ is uniformly bounded by n. Define the sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ by $S_n := T_n \wedge \tau_n$. By construction, we get that P-a.s. S_n tends to infinity and $M^{S_n} := (M_{S_n\wedge t})_{t\geq 0}$ is uniformly bounded by n. Moreover, using the hint, we get that for any n, $M^{S_n} = (M^{T_n})^{\tau_n}$ is a martingale.

d) Let ξ be a r.v. defined on $(\Omega, \mathcal{F}, Q)$ such that $\mathbb{E}[|\xi|] < \infty$ and $E[\xi^2] = \infty$. Consider the process $M = (M_t)_{t\geq 0}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ given by

$$M_t = \begin{cases} 0, & \text{if } 0 \leq t \leq 1, \\ \xi - \mathbb{E}[\xi], & \text{if } t \geq 1, \end{cases} \qquad \mathcal{F}_t = \begin{cases} \{\emptyset, \Omega\}, & \text{if } 0 \leq t \leq 1, \\ \mathcal{F}^\xi = \sigma(\xi), & \text{if } t \geq 1. \end{cases}$$

Then, by construction M is a martingale which is not locally square integrable.

3. **a)** This follows from the Burkholder-Davis-Gundy (BDG) inequalities (cf. Theorem (3.15) in Chapter 4) applied to the continuous local martingale $\int H 1_{]\!]0,\tau]\!]}\, dM$.

b) First, note that the assumption $E\left[\sqrt{\int_0^T H_s^2\, d\langle M\rangle_s}\right] < \infty$ implies $\int_0^T H_s^2\, d\langle M\rangle_s < \infty$ P-a.s. for each $T \geq 0$. Hence, choosing $\tau_n := \inf\{t \geq 0 | \int_0^t H_s^2 d\langle M\rangle_s > n\}$ shows that $H \in L^2_{\text{loc}}(M)$ and $N := \int H\, dM$ is well-defined.

Fix $T \geq 0$. The "right-hand" BDG inequality for $p = 1$ applied to the stopped process N^T gives

$$E\left[\sup_{0\leq t\leq\infty} |N_t^T|\right] \leq C_1 E\left[\sqrt{\langle N^T\rangle_\infty}\right] = C_1 E\left[\sqrt{\int_0^T H_s^2\, d\langle M\rangle_s}\right] < \infty.$$

Being dominated by an integrable random variable, the continuous local martingale N^T is actually a martingale. Since T was arbitrary, is follows that N is a martingale.

c) Fix $T \geq 0$. The "left-hand" BDG inequality for $p = 1$ applied to M^T gives

$$E[\sqrt{\langle M\rangle_T}] = E\left[\sqrt{\langle M^T\rangle_\infty}\right] \leq \frac{1}{c_1} E\left[\sup_{0\leq t\leq\infty} |M_t^T|\right] = \frac{1}{c_1} E\left[\sup_{0\leq t\leq T} |M_t|\right] < \infty.$$

Thus, if $C > 0$ is any constant bounding H^T,

$$E\left[\sqrt{\int_0^T H_s^2\, d\langle M\rangle_s}\right] \leq C E\left[\sqrt{\langle M\rangle_T}\right] < \infty,$$

and part b) applies.

Bitte wenden!

4. Matlab Files

```
function error=bmsc84
% In this exercise we approximate the stochastic integral \int_0^1 W_s dW_s
% and compare it with the exact solution
tic
%% parameter input
% horizon
T=1;
% sample size
Nsimu=10^3;
% grid points
M=10^3;

% Simulate BM with normal increments
BM = [zeros(1,Nsimu);sqrt(T/M)*cumsum(randn(M,Nsimu))];

% Approximate stochastic integral with sum
Int = [zeros(1,Nsimu);cumsum(BM(1:(end-1),:).*(BM(2:end,:)-BM(1:(end-1),:)))];

% Exact solution
timegrid= 0:T/M:T;
Sol = 0.5 *(BM(1:end,:).^2- repmat(timegrid',1,Nsimu));

%plot one sample path
plot(timegrid,Int(:,1),'r-',timegrid,Sol(:,1),'b-')
xlabel('time');
ylabel('value');
legend('sum approximation', 'exact solution');
title('approximation of stochastic integral');

% compute the L^2 error at terminal point: \|(Sum_1-Exact_1)\|_{L^2}
error = mean((Sol(end,:)-Int(end,:)).^2)^(1/2);
toc
```

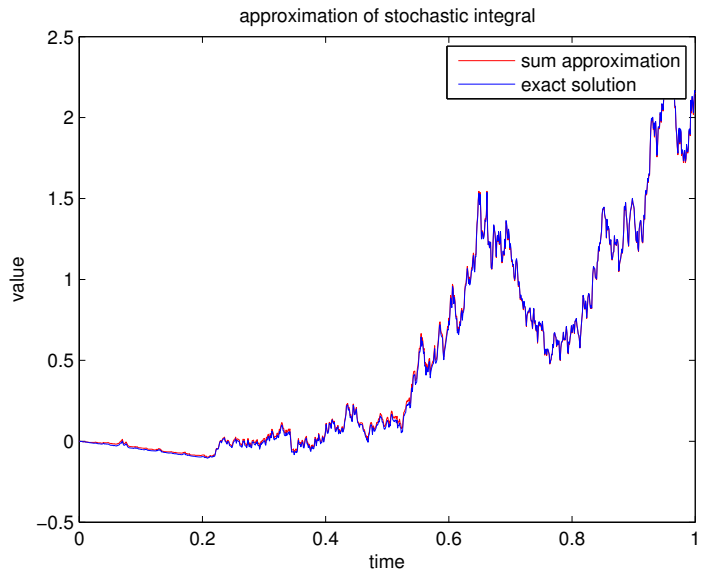

Abbildung 1: One realization of the sum approximation and the exact solution of $\int_0^t B_s dB_s$ on $[0, 1]$.