Brownian Motion and Stochastic Calculus

Sketch of Solution Sheet 9

1. Without loss of generality, assume that $M_0 = 0$. Suppose first that M has variation, denoted by $\mathrm{Var}(M)$, which is uniformly bounded, i.e. assume that

$$\exists K \geq 0 \text{ such that for } P\text{-a.e. } \omega, \quad \forall t \geq 0, \quad \mathrm{Var}_t(M_\cdot(\omega)) \leq K. \tag{1}$$

Fix any $t \geq 0$. Consider a subdivision σ of the interval $[0,t]$ given by: $0 = t_0 < t_1 < \ldots < t_n = t$. We define its mesh size by:

$$\|\sigma\| := \max_{0 \leq i \leq n-1} |t_{i+1} - t_i|.$$

We claim that by the martingale property of M we have for any $0 \leq i \leq n-1$ that

$$E\Big[\big(M_{t_{i+1}} - M_{t_i}\big)^2\Big] = E\Big[M_{t_{i+1}}^2 - M_{t_i}^2\Big]. \tag{2}$$

Indeed, if $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by M, we get by applying the martingale property that

$$\begin{aligned} E\Big[\big(M_{t_{i+1}} - M_{t_i}\big)^2 \,\Big|\, \mathcal{F}_{t_i}\Big] &= E\Big[M_{t_{i+1}}^2 \,\Big|\, \mathcal{F}_{t_i}\Big] - 2M_{t_i}\, E\Big[M_{t_{i+1}} \,\Big|\, \mathcal{F}_{t_i}\Big] + M_{t_i}^2 \\ &= E\Big[M_{t_{i+1}}^2 \,\Big|\, \mathcal{F}_{t_i}\Big] - M_{t_i}^2. \end{aligned}$$

By taking the expectation in the above equality, we proved the claim. Therefore, we deduce from (2) that

$$E\Big[M_t^2\Big] = E\Big[\sum_{i=0}^{n-1} \big(M_{t_{i+1}} - M_{t_i}\big)^2\Big].$$

Thus, due to our assumption (1), we get

$$E\Big[M_t^2\Big] \leq E\Big[\mathrm{Var}_t(M) \max_{0\leq i\leq n-1} |M_{t_{i+1}} - M_{t_i}|\Big] \leq K E\Big[\max_{0\leq i\leq n-1} |M_{t_{i+1}} - M_{t_i}|\Big]. \tag{3}$$

Now, take any sequence $(\sigma_k)_{k\in\mathbb{N}}$ of subdivisions of $[0,t]$ with $\lim\limits_{n\to\infty} \|\sigma_k\| = 0$. Using (3), we deduce from the continuity of M (and so uniform continuity of M on $[0,t]$) and by using domintated convergence theorem, which we can use as $\text{Var}_t(M.(\omega)) \leq K$ for P-a.e. ω by the assumption made in (1), that

$$E\big[M_t^2\big] = 0 \quad \text{which implies that} \quad M_t^2 = 0 \;\; P\text{-a.s.}$$

Since $t \geq 0$ was arbitrarily chosen, we obtain that

$$P\text{-a.s.}, \;\; \forall t \in \mathbb{Q}_+, \;\; M_t = 0.$$

Using the continuity of M we obtain that

$$P\text{-a.s.}, \;\; \forall t \geq 0, \;\; M_t = 0.$$

Now, let M be a continuous martingale of finite variation starting at 0 <u>without</u> satisfying the additional assumption (1). Consider for any $k \in \mathbb{N}$ the stopping time

$$\tau_k := \inf\big\{t \geq 0 \,\big|\, \text{Var}_t(M) \geq k\big\}.$$

As M is an adapted continuous process, Var(M) is continuous and adapted, too. Hence it is easy to check that for any k, τ_k is a stopping time. Moreover, τ_k converges to infinity as k goes to infinity, as M is of finite variation. Moreover, for any k, the stopped process $M_t^{\tau_k} = (M_t^{\tau_k})_{t\geq 0}$ is a continuous martingale of finite variation starting at 0 which satisfies the additional condition (1) (for the constant $K = k$). Thus, from the above result, we obtain that for any $k \in \mathbb{N}$

$$P\text{-a.s.}, \;\; \forall t \geq 0, \;\; M_t^{\tau_k} = 0.$$

Thus, letting k goes to infinity, we obtain the desired result.

2. **a)** By linearity, it suffices to check the claim for monomials of the form $p(x) = x^m, m \in \mathbb{N}$. Note that $p(W)$ is (left-)continuous and adapted, (and hence predictable and locally bounded). Therefore, $\int p(W)dW$ is well-defined, and also a local martingale. Moreover, by Fubini's Theorem, for all $T \geq 0$,

$$\mathbb{E}\left[\left\langle \int p(W)dW \right\rangle_T\right] = E\Big[\int_0^T W_s^{2m} d\langle W\rangle_s\Big] \tag{4}$$

$$= E\Big[\int_0^T W_s^{2m} ds\Big] \tag{5}$$

$$= \int_0^T E\Big[W_s^{2m}\Big] ds \tag{6}$$

$$= E[W_1^{2m}] \int_0^T s^m ds < \infty. \tag{7}$$

This proves that $\left(\int p(W)dW\right)^T \in \mathcal{H}_0^{2,c}$ for all $T \geq 0$ by Ex 7-2 a), implying that $\int p(W)dW$ is a true martingale.

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b) The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(t,w) := e^{\frac{1}{2}t}\cos w$ is C^2 and $X_t = f(t, W_t)$. Moreover,

$$\frac{\partial f}{\partial t}(t,w) = \frac{1}{2}e^{\frac{1}{2}t}\cos w,\ \frac{\partial f}{\partial w}(t,w) = -e^{\frac{1}{2}t}\sin w,\ \frac{\partial^2 f}{\partial w^2}(t,w) = -e^{\frac{1}{2}t}\cos w.$$

Since t (viewed as a process) is of finite variation, Itô's formula yields

$$\begin{aligned} dX_t &= \frac{\partial f}{\partial t}(t,w)\,dt + \frac{\partial f}{\partial w}(t,w)\,dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(t,w)\,d\langle W]_t \\ &= -e^{\frac{1}{2}t}\sin W_t\,dW_t, \end{aligned}$$

so X is a local martingale. Since $\sup_{0\le t\le T}|X_t| \le e^{\frac{1}{2}T}$ for each $T \ge 0$, X is a martingale.

c) Being adapted, left-continuous and bounded, $\varrho \in L^2_{\text{loc}}(W)$ and $\sqrt{1-\varrho^2} \in L^2_{\text{loc}}(W')$. Moreover, for each $t \ge 0$, using bilinearity of $[\cdot,\cdot]$ and the fact that $[W, W'] = 0$ due to independence of W and W',

$$[B]_t = \left[\int \varrho\,dW\right]_t + \left[\int \sqrt{1-\varrho^2}\,dW'\right]_t = \int_0^t \varrho_s^2\,ds + \int_0^t (1-\varrho_s^2)\,ds = t,$$

so Lévy's characterisation of Brownian motion yields that B is a Brownian motion. Finally,

$$[B, W\rangle_t = \int_0^t \varrho_s\,d[W, W\rangle_s = \int_0^t \varrho_s\,ds.$$

3. **a)** Let $t > 0$. Using that ΔN_t is either 0 or 1, we have P-a.s.

$$\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{\Delta N_t} = \frac{\widetilde{\lambda}}{\lambda}\Delta N_t + (1 - \Delta N_t) = 1 + \frac{\widetilde{\lambda} - \lambda}{\lambda}\Delta N_t\,.$$

Using this, we arrive at

$$\begin{aligned} S_t = e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_{t-}+\Delta N_t} \Rightarrow \Delta S_t &= e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_{t-}}\left(\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{\Delta N_t} - 1\right) \\ &= S_{t-}\frac{\widetilde{\lambda} - \lambda}{\lambda}\Delta N_t\,. \end{aligned} \tag{8}$$

b) We have

$$S_t = \exp\left((\lambda - \widetilde{\lambda})t + \log(\widetilde{\lambda}/\lambda)N_t\right). \tag{9}$$

Applying the hint with $f(\cdot) = \exp(\cdot)$, $\alpha = \lambda - \widetilde{\lambda}$, $\beta = \log(\widetilde{\lambda}/\lambda)$ and using part a) and $X_t = \log(\widetilde{\lambda}/\lambda)N_t + (\lambda - \widetilde{\lambda})t$, we get P- a.s. for all $t \geq 0$

$$\begin{aligned}
S_t &= \exp(0) + (\lambda - \widetilde{\lambda}) \int_0^t \exp(X_{u-})\,\mathrm{d}u + \sum_{0<u\leq t} \Big(\exp(X_u) - \exp(X_{u-})\Big) \\
&= 1 + (\lambda - \widetilde{\lambda}) \int_0^t S_{u-}\,\mathrm{d}u + \sum_{0<u\leq t} \Delta S_u \\
&= 1 + \frac{\widetilde{\lambda} - \lambda}{\lambda}\Big(- \int_0^t S_{u-}\lambda du + \sum_{0<u\leq t} S_{u-}\Delta N_u \Big) \\
&= 1 + \frac{\widetilde{\lambda} - \lambda}{\lambda} \int_0^t S_{u-}(\,\mathrm{d}N_u - \,\mathrm{d}(\lambda u)) = 1 + \frac{\widetilde{\lambda} - \lambda}{\lambda} \int_0^t S_{u-}\,\mathrm{d}\widetilde{N}_u\,.
\end{aligned}$$

c) It can be easily verified that $\widetilde{N}$ is a $(P, \mathcal{F})$-martingale. Since $\frac{\widetilde{\lambda}-\lambda}{\lambda} S_-$ is adapted and left-continuous, (hence predictable and locally bounded), it follows that S is a local $(P, \mathcal{F})$-martingale. By the hint, S is a true $(P, \mathcal{F})$-martingale if

$$\mathbb{E}[\sup_{0\leq t\leq T} |S_t|] = \mathbb{E}[\sup_{0\leq t\leq T} S_t] < \infty. \tag{10}$$

But since

$$\sup_{0\leq t\leq T} S_t \leq Ce^{N_T}$$

for some constant $C > 0$ and since $N_T \sim$ Pois(λT), we conclude that (10) is true.

4. **a)** Let $X = (X_t)_{t\geq 0}$ be a uniformly integrable, right-continuous martingale. Set $Y := Z(X^\tau - X^\sigma)$ and fix a stopping time ϱ. We will show that $E[|Y_\varrho|] < \infty$ and $E[Y_\varrho] = 0$. The assertion then follows from the hint (cf. Lemma 4.1.19 in the lecture notes).

Since X is uniformly integrable, the stopping theorem yields $E[X_\infty|\mathcal{F}_\gamma] = X_\gamma$ for any stopping time γ. In particular, the family $\{X_\gamma : \gamma \text{ a stopping time}\}$ is uniformly integrable (i.e., X is of *class (D)*), hence bounded in L^1. It follows that

$$E[|Y_\varrho|] \leq C(E[|X_{\tau\wedge\varrho}|] + E[|X_{\sigma\wedge\varrho}|]) < \infty$$

where $C > 0$ is any constant bounding Z.

Next, we show that $E[Y_\varrho] = 0$. By a monotone class argument or simply measure-theoretic induction, we may assume that $Z = 1_A$ for some $A \in \mathcal{F}_\sigma$. Then $\tau_A := \tau 1_A + \infty 1_{A^c}$ and $\sigma_A := \sigma 1_A + \infty 1_{A^c}$ are stopping times and

$$E[Y_\varrho] = E[1_A(X_{\varrho\wedge\tau} - X_{\varrho\wedge\sigma})] = E[X_{\varrho\wedge\tau_A} - X_{\varrho\wedge\sigma_A}] = 0,$$

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where we use the stopping theorem in the last equality.

If X is not uniformly integrable, then assuming that ϱ is bounded, almost the same proof yields that Y is a martingale (but not uniformly integrable in general), c.f. Remark 4.(1.20) in the lecture notes.

b) The equality $B := Z[M^\tau - M^\sigma, N] = Z([M,N]^\tau - [M,N]^\sigma)$ follows from bilinearity of $[\cdot,\cdot]$ and from the fact that for any stopping time τ,

$$[M^\tau, N] = [M, N^\tau] = [M,N]^\tau.$$

Next, we note that $Y := Z(M^\tau - M^\sigma) \in \mathcal{M}^c_{0,\mathrm{loc}}$ by part **a)** and localisation. So $[Y,N]$ is well-defined. We also note that the process B is continuous and of finite variation. Moreover, since $B = 0$ on $[\![0,\sigma]\!]$, we can write $B = (Z1_{]\!]\sigma,\infty]\!]})([M,N]^\tau - [M,N]^\sigma)$ to see that B is also adapted.

Setting $X := (M^\tau - M^\sigma)N - [M^\tau - M^\sigma, N] \in \mathcal{M}^c_{0,\mathrm{loc}}$ and noting that $X^\sigma = 0$, we have

$$YN - B = Z((M^\tau - M^\sigma)N - [M^\tau - M^\sigma, N]) = Z(X - X^\sigma).$$

By part **a)** and localisation, $Z(X - X^\sigma) \in \mathcal{M}^c_{0,\mathrm{loc}}$. Thus, as $[Y,N]$ is the unique process $\widetilde{B}$ of cFV_0 such that $MN - \widetilde{B} \in \mathcal{M}^c_{0,loc}$, we conclude by uniqueness that $[Y,N] = B$.

c) Clearly, $H := Z1_{]\!]\sigma,\tau]\!]}$ is left-continuous. Moreover for $t \geq 0$, the second factor in $H_t = (Z1_{\{\sigma<t\}})1_{\{t\leq\tau\}}$ is $\mathcal{F}_t$-measurable since τ is a stopping time, while the $\mathcal{F}_t$-measurability of the first factor follows from the hint. Thus, H is adapted and hence predictable. Since H is also bounded, it follows that the stochastic integral is well-defined. Now, for any $N \in \mathcal{M}^c_{0,\mathrm{loc}}$, we have

$$[Z(M^\tau - M^\sigma), N] \overset{\text{b)}}{=} Z([M,N]^\tau - [M,N]^\sigma) = \int Z1_{]\!]\sigma,\tau]\!]}\, d[M,N] = \int H d[M,N].$$

Thus by the defining property of the stochastic integral, $\int H\, dM = Z(M^\tau - M^\sigma)$ (cf. Proposition 4.2.16 in the lecture notes.)

Finally, from part **a)**, we see that if M is a (uniformly integrable) martingale, then $\int H\, dM = Z(M^\tau - M^\sigma)$ is a (uniformly integrable) martingale.