Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 1

1. (a) Let Γ be a group acting discretely and properly discontinuously on (M, g) by isometries. Observe that M/Γ is a smooth manifold and M/Γ inherits a Riemannian metric from M such that $\pi\colon M \to M/\Gamma$ is locally an isometry.

(b) Find an isometric immersion of the unit square torus $\mathbb{R}^2/\mathbb{Z}^2$ into $\mathbb{R}^4$.

(c) Let v, w be a basis of $\mathbb{R}^2$ and let $\Gamma := \mathbb{Z} \cdot v + \mathbb{Z} \cdot w$ be the lattice they generate. Then $\mathbb{R}^2/\Gamma$ is a torus with a locally Euclidean metric. Show that not all $\mathbb{R}^2/\Gamma$ are isometric, even up to scale.

(d) Can you visualize the set of isometry classes of *all* such tori $\mathbb{R}^2/\Gamma$, say of area 1?

2. (a) The *upper half-plane model* of the hyperbolic plane is the set

$$\mathbb{H}^2 := \{z \in \mathbb{C} | \Im(z) > 0\}$$

equipped with the metric

$$g_{ij}(z) = \frac{\delta_{ij}}{y^2}, \quad z = x + iy \in H.$$

Prove that each *fractional linear transformation*

$$z \mapsto \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{R}$$

is an isometry of g. Hint: Show that for f holomorphic and $g_{ij} = \lambda(z)\delta_{ij}$,

$$f^*(g)(z) = |f'(z)|^2 \lambda(f(z))\delta_{ij}.$$

(b) Find an isometry of the upper half-plane model with the *disk model*

$$h_{ij}(z) = \frac{4\delta_{ij}}{(1 - |z|^2)^2}, \quad |z| < 1.$$

Hint: Try a fractional linear transformation with complex coefficients.

(c) Show that the hyperbolic plane is homogenous and isotropic.

3. Let $B_r^{\mathbb{H}^2}$ be a ball of (intrinsic) radius r in the hzperbolic plane.

(a) compute the circumference $C(r)$ and the area $A(r)$ of $B_r^{\mathbb{H}^2}$

(b) Check that $\frac{dA(r)}{dr} = C(r)$. Why should this be so?

4. (a) Let $L : V \to W$ be a linear map between inner product spaces. Prove: there is orthonormal basis $v_1 \ldots, v_n$, $w_1 \ldots, w_m$ for V and resp. W and *singular values* (or *principal stretches*) $\lambda_1, \ldots, \lambda_k \geqslant 0$, $k = \min(m, n)$ such that $Lv_i = \lambda_i w_i$ for $i = 1, \ldots k$.

(b) Prove the *singular value decomposition* from linear algebra: for all $A \in M^{n\times n}(\mathbb{R})$ there exist matrices $O, O' \in O(n)$ and a diagonal matrix D such that $A = ODO'$.

(c) Prove the *polar decomposition*: for all $A \in M^{n\times n}(\mathbb{R})$ there exists $O \in O(n)$, S symmetric such that $A = SO$.

5. (a) Show the metric induced on $SO(n)$ by the inclusion in $\mathbb{R}^{n\times n} = \mathbb{R}^{n^2}$ is bi-invariant (i.e both left-invariant and right-invariant)

(b) For $a \in G$, let $AD_a : G \to G$ be defined by $AD_a(b) := aba^{-1}$, and let $Ad_a : T_eG \to T_eG$ be defined by $Ad_a := d(AD_a)_e$. Verify that $Ad : G \to GL(\mathcal{G})$ is a homomorphism. It is called the *adjoint representation* of G on $\mathcal{G}$.

(c) Prove: if G has a bi-invariant measure, then G has a bi-invariant metric. Hint: Let $h(e)$ be any metric on T_eG. Average $(Ad_a)^*(h(e))$ over the group G to get an Ad-invariant metric $g(e)$ on T_eG. Now extend $g(e)$ to a left-invariant metric g on G and verify that g is bi-invariant.

Remark: Every compact Lie group has a bi-invariant measure (called Haar measure, see Lee, p. 46, problem 3–11) and hence a bi-invariant metric.

Due on Friday March 4