Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 10

1. Let M be a connected manifold. Two metrics g and $\tilde{g}$ on M are called *conformally equivalent* if there is a smooth, positive function ϕ on M such that $\tilde{g} = \phi g$. The metrics g and $\tilde{g}$ then have the same angles but not the same lengths (unless $\phi \equiv 1$).

Let g fixed.

(a) Show that there is a complete metric $\tilde{g}$ on M that is conformally equivalent to g.

(b) Show that there is a conformally equivalent metric $\tilde{g}$ on M of finite diameter. (N.B. $\operatorname{diam}(M, g) := \sup\{d(x, y) \ : \ x, y \in M\}$).

(c) Observe that for any given compact subset $K \subset M$, the conformally equivalent metric $\tilde{g}$ in **a)** or **b)** may be chosen such that $\tilde{g}|_K = g|_K$.

2. Construct a regular octagon in the hyperbolic plane H^2 as follows. Let $\tilde{p_j} := re^{2\pi ij/8}$, $j = 0, 1, 2, \ldots, 7$, be the vertices of a regular octagon in T_qH^2, let $p_j := \exp_q(\tilde{p_j})$, and let γ_j be the (unique) geodesic segment in H^2 connecting p_j to p_{j+1}. P_r is the region bounded by $\bigcup_{j=0}^{7} \gamma_j$.

(N.B: $\exp_p^{-1}$ is not a polygon but has a spiky look).

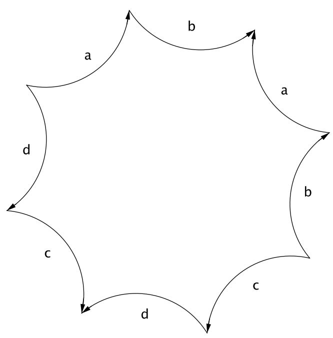
(a) Glue the edges of P_r to each other as depicted. How many edges and vertices does the glued figure have?

(b) Show that for some $r > 0$, the construction yields a smooth, closed surface Σ that is locally isometric to the hyperbolic plane. *Hint:* study the interior angles of P_r as a function of r. The case $r \to 0$ and $r \to \infty$ are easier.

(c) What is the genus and Euler characteristic of Σ?

(d) (*) How many isometries does Σ have?

(e) (**) Find the diameter of Σ.



3. Let V and W be real vector spaces. A tensor $T \in V \otimes W$ is called *decomposable* or a *pure* tensor if $T = v \otimes w$ for some vectors $v \in V$ and $w \in W$. Let P be the pure tensors in $V \otimes W$.

(a) Find an algebraic criterion that selects the pure tensors in $\mathbb{R}^2 \otimes \mathbb{R}^2$. Describe this set geometrically.

(b) Show (for general V and W) that P is a *quadratic variety* and a cone.

(c) Show P is a smooth manifold except at the origin and calculate the dimension.

4. *(Symmetric Spaces)* Let M be a connected Riemannian manifold such that for each $p \in M$ there is an isometry σ_p of M such that

i) σ_p fixes p

ii) σ_p reverses each geodesic through p.

Such an M is called a *symmetric space.*

(a) Show that the maps σ_p satisfying i) and ii) are unique.

(b) Show that the map

$$M \times M \to M, \qquad (p, q) \mapsto \sigma_p(q),$$

is smooth.

(c) Show that M is complete.

(d) Let γ be a non-constant geodesic. Define

$$\tau_{\gamma,t} = \sigma_{\gamma(t/2)} \circ \sigma_{\gamma(0)}.$$

Show that $t \mapsto \tau_{\gamma,t}$ is a one-parameter group of isometries that 'translates' along γ.

5. (a) Prove that $\mathbb{H}^n = \left(B, \frac{4\sum_{i=1}^n (dx^i)^n}{(1-|x|^2)}\right)$ is a symmetric space by exhibiting the isometry σ_x, $x \in \mathbb{H}^n$.

(b) Let $S_+ = \left\{g \in Sym^2(\mathbb{R}^n) \mid g > 0\right\}$ the space of positive symmetric tensors. Give S_+ the Riemannian metric G

$$G_g(h, f) = \langle h, l\rangle_g = g^{ij} g^{kl} h_{i,j} f_{j,l}$$

where $h, f \in T_g S_+$. Show (S_+, G) is asymmetric space.

(c) If $n = 2$, show that S_+ is isometric to $\mathbb{H} \times \mathbb{R}$.

Hint: it helps to define

$$S_+^1 = \{g \in S_+ \mid \det(g)_\delta = 1\}$$

where $\delta = (\delta_{ij})$ and $\det_h(g) = \det(h^{-1}g)$. Then show that $PSL_2(\mathbb{R})$ acts on S_+^1.

Due on Friday 15 May