Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 11

1. Let M be a Riemannian manifold. Show that in local coordinates $x^1, \ldots, x^n$ the components of the curvature tensor $R^i{}_{jkl} = -dx^i\left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}\right)$ are given by

$$R^i{}_{jkl} = -\frac{\partial}{\partial x^k}\Gamma^i{}_{lj} + \frac{\partial}{\partial x^l}\Gamma^i{}_{kj} - \Gamma^i{}_{kp}\Gamma^p{}_{lj} + \Gamma^i{}_{lp}\Gamma^p{}_{kj}$$

with the Christoffel symbols $\Gamma^i{}_{jk} = dx^i\left(D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}\right)$.

2. Show that on a Lie group with a bi-invariant metric

(a)

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z] \text{ and } \mathrm{Rm}(X,Y,X,Y) = \frac{1}{4}\left|[X,Y]\right|^2,$$

for all left-invariant vector fields X, Y and Z.

(b) Let $q \in M$ and P be a 2-plane in T_qM. The *sectional curvature* of P is defined by

$$k(q,P) := Rm(e_1, e_2, e_1, e_2)$$

where e_1, e_2 is an ortonormal basis of P. (We will see in the next exercise sheet that the sectional curvature is well-defined). Compute the sectional curvature of S^3 (use the bi -invariant metric of Exercise sheet 2).

(c) Show that the sectional curvature of $SO(n)$ is everywhere non-negative (use the bi -invariant metric of Exercise sheet 1).

3. *(Infinitesimal Circumference and Area)* Let M be a Riemannian 2-manifold. Let $C(r)$ and $A(r)$ denote the circumference and area of the geodesic ball $B_r(p)$ in M, and set $K = K(p) = R(e_1, e_2, e_1, e_2)$. Show

$$C(r) = 2\pi\left(r - \frac{Kr^3}{6} + O(r^4)\right),$$

$$A(r) = \pi\left(r^2 - \frac{Kr^4}{12} + O(r^5)\right).$$

4. *(Theorema Egregium)* Let M be isometrically embedded in $\mathbb{R}^3$. Show

$$K = k_1 k_2,$$

where k_1 and k_2 are the principle curvatures. *Hint:* Write M as a graph over T_pM of a function u with $u(0) = 0$ and $Du(0) = 0$. Then k_1 and k_2 are the eigenvalues of $D^2u(0)$. Now Exercise 1 and/or Exercise 3 could be helpful.

5. *(Symmetric Spaces)* Let M be a connected Riemannian manifold. Assume that M is a symmetric space.

(a) Show that for each vector V in T_pM there is a unique Killing field X on M such that $X(p) = V$, $DX(p) = 0$.

(b) Compute the Riemann curvature at p using these Killing fields.

Due on Friday 22 May