Differential Geometry II

D-MATH, FS 2015

Tom Ilmanen

Exercise Sheet 12

1. *(Space of Curvature Tensors)* What is the dimension of the space of potential Riemann curvature tensors at a point? That is: what is the dimension of the vector space of 4-linear maps $T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$ satisfying the symmetries $R_{ijkl} = -R_{jikl} = R_{klij}$, $R_{ijkl} + R_{jkil} + R_{kijl} = 0$?

2. *(Curvature of a 2-Manifold)* Let M be a Riemannian 2-manifold.

(a) Show that the Riemann curvature tensor of M is effectively a scalar. Namely, we have

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}$$

and the remaining components vanish.

(b) Show that

$$K(p) = Rm(e_1, e_2, e_1, e_2)$$

is independent of the choice of orthonormal basis e_1, e_2 of T_pM.

(c) How can we reconstruct the tensor $R_{ijkl}(p)$ from $K(p)$?

3. (*Curvature of Hyperbolic Space*) Compute the Riemann curvature tensor, sectional, Ricci and scalar curvatures of hyperbolic space H^n. Note that hyperbolic space can be defined via the *ball model*

$$g_{ij}(x) = \frac{4\delta_{ij}}{(1-|x|^2)^2}, \qquad (x^1)^2 + \cdots + (x^n)^2 < 1,$$

or equivalently the *upper-half space model*

$$g_{ij}(x) = \frac{\delta_{ij}}{(x^n)^2}, \qquad x^n > 0.$$

4. (*Compact Semisimple Lie Groups*) Let G be a compact Lie Group with Lie algebra $\mathfrak{g}$. Define the *Killing form* of G by

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad B(X,Y) := \mathrm{tr}(\mathrm{Ad}(X) \circ \mathrm{Ad}(Y)).$$

(a) Show that B is bilinear, symmetric and Ad-invariant.

(b) If G is compact, show that B is nonpositive. So if B is negative, then $g := -B$ is a canonical choice of metric on G — meaning that it is defined from the group structure alone.

Hint: Compute the trace by summiwng over an orthonormal basis (with respect to any bi-invariant metric on G).

(c) Show that for a compact, semisimple Lie group with this canonical metric,

$$\mathrm{Rc} = \frac{1}{4} g$$

where Rc denotes the Ricci curvature of g.

Note: A Lie group whose Killing form is nondegenerate is called *semisimple.* A manifold whose Ricci curvature is proportional to its metric is called *Einstein.* So a compact semisimple Lie group is Einstein.

d) (*) What class of compact Lie groups is excluded by requiring G to be semisimple? How much freedom do we have to choose a bi-invariant metric on a compact Lie group, if we do not use the Killing form?

Due on Friday 29 May