Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 2

1. (a) Prove that a Möbious transformation $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$

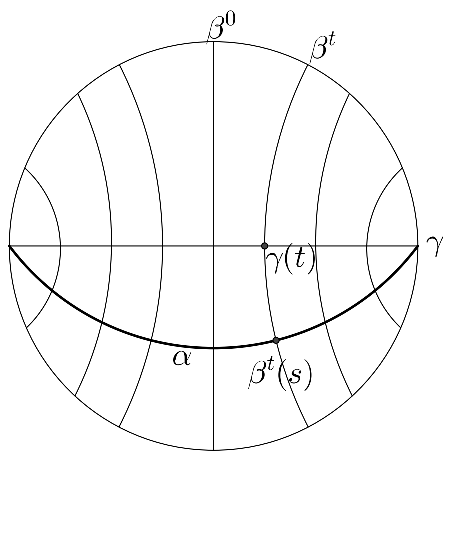
$$T(z) := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1$$

preserves generalized lines (i.e circles and lines).

(b) Determine the group of fractional linear transformations that preserves the unit disk B^2.

(c) Let $\mathbb{H}^2 = \left(B^2, \frac{4\delta}{(1-|z|^2)^2}\right)$ be the Poincare disk model of the hyperbolic plane. Prove : the geodesics are precisely the arcs of circles and line segment in B^2, that are perpendicular to ∂B^2.
Hint: you may use the results of ex. 1 of Supplementary Exercises and ex. 2 of Exercise Sheet 1.

2. Let $\gamma(t), t \in \mathbb{R}$ be a geodesic in $\mathbb{H}^2$, parametrized by arclength. For each $t \in \mathbb{R}$ let β^t be the geodesic in $\mathbb{H}^2$ that is perpendicular to γ at the point $\gamma(t)$. As t varies, the geodesic of β^t sweep out all of $\mathbb{H}^2$.
Now parametrized β^t by arclength, so that $\beta^t(s)$ has distance s to γ. Fix s and consider the moving point $t \to \alpha(t) := \beta^t(s), t \in \mathbb{R}$.



Show

(a) $\alpha(t)$ maintains a constant distance s to γ, but $\alpha(t)$ is not a geodesic;

(b) $\dot{\alpha(t)}$ is orthogonal to β^t;

(c) Compute the speed of $\dot{\alpha(t)}$ as a function of s.

3. Let S^3 be the unit quaternions. Recall the left invariant vectorfields I, J, K on S^3 defined by

$$I(u) := ui, \quad J(u) := uj, \quad K(u) := uk, \quad u \in S^3$$

For $\lambda > 0$ we define a Riemannian metric on S^3 by requiring that I, J, K be orthogonal for g_λ and

$$g_\lambda(I, I) = \lambda^2, \quad g_\lambda(J, J) = g_\lambda(K, K) = 1.$$

(a) Verify that g_λ is left-invariant.

(b) Verify that g_1 is bi-invariant and is the standard metric on S^3.

4. We consider two degenerate situations.

(a) Describe geometric how g_λ looks as λ go to 0. Hint: Use the Hopf fibration $S^3 \to S^2$.

(b) What happens as λ go to ∞? Obviously the distance function

$$d_\lambda(x, y) = \mathrm{dist}_{g_\lambda}(x, y)$$

is increasing in λ but the limit is not given by a Riemannian metric. Prove:

$$d_\infty(x, y) := \lim_{\lambda \to \infty} \mathrm{dist}_{g_\lambda}(x, y)$$

exists and defines a metric (in the sense of metric spaces).
Hint: Show that any two point $x, y \in S^3$ can be connected by a path that is tangent to the 2-plane field spanned by $J(z), K(z)$ at each point $z \in S^3$.

Due on Friday March 13