Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 4

1. Define the oriented 2-plane bundle E_k over S^2 by gluing $B_1 \times \mathbb{R}^2$ to $B_1 \times \mathbb{R}^2$ via the map

$$\begin{array}{llcl} \phi_k: & \partial B_1 \times \mathbb{R}^2 & \to & \partial B_1 \times \mathbb{R}^2 \\ & (e^{i\theta}, (x,y)) & \mapsto & (e^{-i\theta}, R_{-k\theta}(x,y)) \end{array}$$

where $R_{-k\theta}$ is rotation by $-k\theta$, and endowing the result with the obvious orientation (this definition is the same with correct signs). The *tautological bundle* over $\mathbb{C}P^N$ is a complex line bundle defined by associating to each point $L \in \mathbb{C}P^N$ the complex line $L \subseteq \mathbb{C}^{N+1}$, i.e. the total space is

$$F_N := \bigcup_{L \in \mathbb{C}P^N} \{L\} \times L$$

with an appropriate smooth structure. F_1 may be viewed as a real 2-plane bundle over $\mathbb{C}P^1 \cong S^2$. What is its twisting number?

2. We have previously seen that Lie derivatives $L_X Y$ can be used to define "derivatives of vector fields".

(a) Show that the map $L\colon TM \times TM \to TM$ is not a connection.

(b) Show that there are vector fields X, Y on $\mathbb{R}^2$ such that Y is constant along the x^1-axis, X is $\frac{\partial}{\partial x^1}$ along the x^1-axis, yet $L_X Y$ does not vanish along the x^1-axis. (This shows that Lie differentiation does not give a reasonable way to take directional derivatives of vector fields along curves.)

3. Let $E \to M$ be a vector bundle with an inner product h on it and a connection D^E compatible with h. Let $F \subseteq E$ be a subbundle and let $\pi_F\colon E \to F$ be the orthogonal projection, i.e. $\pi_F(x)\colon E_x \to F_x$ is orthogonal projection for each $x \in M$. Let $X \in C^\infty(TM)$, $V \in C^\infty(F)$. Define

$$D^F_X V := \pi_F(D^E_X V).$$

Show that D^F is a connection on F and that D^F is compatible with the metric on F induced by restricting h to F.

4. Let D be a connection on a vector bundle E that is compatible with the inner product $\langle \cdot, \cdot \rangle$ on E.

(a) Show: if V is a parallel section of E, then V has constant length.

(b) Let $e_1, \cdots, e_d$ be an orthonormal frame for E over U. Write

$$D = D^0 + \Delta$$

where D^0 is the frame connection and $\Delta = \Delta(X, V) =: \Delta_X Y$. Show that

$$\Delta_X : E \to E$$

is antisymmertic with respect to $\langle \cdot, \cdot \rangle$, i.e.

$$\langle \Delta_X V, W \rangle + \langle V, \Delta_X W \rangle = 0.$$

In components, this reads $(\Delta_X)_\alpha^\beta + (\Delta_X)_\beta^\alpha = 0$.

Due on Friday March 27