Differential Geometry II

**D-MATH, FS 2015**

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# Exercise Sheet 5

**1.** Let $D^1, D^2$ be two connections on a vector bundle $E$.

(a) Show
$$\Delta(X, V) := D^2_X V - D^1_X V$$
defines a bilinear map
$$\Delta(p)\colon T_pM \times E_p \to E_p$$
at each point $p \in M$. (That is, $\Delta(X, Y)(p)$ depends only on $X(p)$ and $V(p)$ and not on derivatives of $X$ or $V$ at $p$.) Note that $\Delta$ defines a section of the bundle Bilin $(TM, E; E)$.

(b) Show for any connection $D^1$ on $E$ and any smooth family of bilinear maps $\Delta(p)\colon T_pM \times E_p \to E_p$, the expression
$$D^2_X V := D^1_X V + \Delta(X, V)$$
defines a connection on $E$.

(c) Conclude that the space of (smooth) connections on $E$ is an affine space over the vectorspace $C^\infty$ (Bilin $(TM, E; E)$).

**2.** Let $E$ be a trivial line bundle over $M$. Let $\langle -, - \rangle_E$ be an inner product on the fibers of $E$. Suppose $D$ is a connection on $E$ that is compatible with $\langle -, - \rangle_E$. Prove that there exists a parallel section for $D$.

**3.** Consider the upper half-plane
$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$
equipped with the hyperbolic metric
$$g_{ij} := \frac{\delta_{ij}}{y^2}.$$

(a) Show that the Christoffel symbols of the Riemannian connection of $g$ are
$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}.$$

(b) Let $Y_0 := (1, 0)$ be a tangent vector at the point $(0, 1) \in \mathbb{R}^2_+$. Let $Y(t)$ be the parallel transport of $Y_0$ along the curve $\gamma\colon t \mapsto (t, 1)$. Show that $Y(t)$ forms an angle $-t$ with $\gamma$. Draw it

(c) Conclude heuristically that a traveler moving along $\gamma$ is turning to the left.

**4.** Let $E$ be a complex line bundle over $M = \mathbb{R}^2$ equipped with an $i$-invariant inner product $\langle \cdot, \cdot \rangle$, i.e. $\langle iX, iY \rangle = \langle X, Y \rangle$ for all $X, Y$, and let $D$ be a connection compatible with $\langle \cdot, \cdot \rangle$.

(a) Show that as a real 2-plane bundle, $E$ possesses a global orthonormal frame $e_1, e_2$ satisfying $e_2 = ie_1$. (Recall that any vector bundle over a contractible space is trivial.)

(b) Show that $D$ has the form

$$D = D^0 - i\omega,$$

where $\omega$ is a section of $C^\infty(T^*M)$ (a 1-form), $i\colon E_p \to E_p$ is multiplication by the complex unit, and $D^0$ is the connection induced by the frame $e_1, e_2$. Conversely, any operator $D$ of this form is a connection on $E$ compatible with $\langle \cdot, \cdot \rangle$. (The $-$sign is only an useful convention.)

(c) Show that $D$ is also *compatible with* $i$, that is

$$D_X(iV) = iD_X V$$

for all $X \in C^\infty(TM), V \in C^\infty(E)$.

(d) Let $V$ be a section of the form $V = e^{i\theta}e_1$, $\theta \in C^\infty(\mathbb{R}^2)$. Show: $V$ is parallel if and only if $d\theta = \omega$.

(e) Write the 1-form $\omega$ as

$$\omega = a(x, y)\, dx + b(x, y)\, dy \quad \text{on } M = \mathbb{R}^2,$$

$a$, $b \in C^\infty(M)$. Show that $E$ possesses a parallel section if and only if

$$\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = 0.$$

Since $a$ and $b$ can be specified arbitrarily, it is not very likely that a random chosen connection $D$ has a parallel section.

**5.** Let $(M, g)$ be a Riemannian manifold and let $(N, h)$ be an isometrically embedded submanifold. Show that the Levi-Civita connection of $(N, h)$ is obtained from the Levi-Civita connection of $(M, g)$ by orthogonal projection.

**Due on Friday April 3**