Differential Geometry II
D-MATH, FS 2015

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Exercise Sheet 6

1. Let $E \to \mathbb{R}^2, D = D^0 - i\omega, \langle\cdot,\cdot\rangle$ be as in sheet 5, $\omega \in C^\infty(TM^*)$. Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a closed curve, $\gamma(0) = \gamma(1) = p$.

(a) Show that the holonomy $H_{p,\gamma}\colon E_p \to E_p$ is given by

$$H_{p,\gamma}(V) = e^{i\int_0^1 \omega(\gamma(t))(\dot\gamma(t))dt}V, \quad V \in E_p.$$

(b) If γ is the boundary of a region $W \subseteq \mathbb{R}^2$, and D is defined over W, then convert the above expression to an integral over W:

$$H_{p,\gamma}(V) = e^{i\int_W \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)dxdy}V,$$

where $\omega = a(x,y)\,dx + b(x,y)\,dy$.

(c) Now work over $U := \mathbb{R}^2\backslash\{0\}$, and define ω on U by

$$\omega := \lambda\frac{-x\,dy + y\,dx}{x^2+y^2}, \quad (x,y) \neq (0,0),$$

where $\lambda \in \mathbb{R}$ is constant. Show that E possesses a parallel section in a small ball around each point of U.

(d) For what values of λ does $E|_U$ possess a global parallel section?

2. Consider the surface of revolution $M \subseteq \mathbb{R}^3$ parametrized by

$$(x,\theta) \mapsto (x, u(x)\cos(\theta), u(x)\sin(\theta))$$

where $u : \mathbb{R} \to (0,\infty)$ is smooth.

(a) Compute the Christoffel symbols of the induced metric in (x,θ) coordinates.

(b) Show that each longitude $\{\theta = \theta_0\}$ is a geodesic in M.

(c) Derive necessary and sufficient conditions for a latitude circle $\{x = x_0\}$ to be a geodesic.

3. (a) Let M be a submanifold of $\mathbb{R}^n$. Show that a curve γ in M is a geodesic if and only if its acceleration (in $\mathbb{R}^n$) is perpendicular to M at every point.

(b) Let M be a surface in $\mathbb{R}^3$ and P a plane that intersects M orthogonally. Show that $\gamma := M \cap P$ is the trace of a geodesic in M.

(c) Use this to find some easy geodesics on a surface of revolution.

4. (a) Consider the 1-parameter family of block matrices

$$A(t) := \begin{pmatrix} R(\alpha_1 t) & & & \\ & \ddots & & \\ & & R(\alpha_m t) & \\ & & & 1 \end{pmatrix} \in \mathrm{SO}(n)$$

where

$$R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$m = \lfloor n/2 \rfloor$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and 1 appears if and only if n is odd.

Show that the $A(t)$'s are geodesics in $\mathrm{SO}(n) \subset \mathbb{R}^{n\times n}$.

(b) Show that every geodesic in $\mathrm{SO}(n)$ starting at the identity matrix has the above form for some choice of basis of $\mathbb{R}^n$.

5. For $x, y \in \mathbb{R}$, $y > 0$, let $f_{x,y}$ denote the affine transformation

$$f_{x,y}(t) := yt + x, \qquad t \in \mathbb{R},$$

of the real line. Let G be the set of all such transformations.

(a) Observe that, under composition, G is a Lie group whose underlying manifold is the upper half plane $\mathbb{R} \times \mathbb{R}_+$ and whose identity element is $(0, 1)$.

(b) Show that G has no bi-invariant metric.

(c) Determine a left-invariant metric g on G via

$$g_{ij}(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

What familiar Riemannian manifold is this?

(d) Show that the geodesics that pass through the identity element of G are *not* one-parameter subgroups. So the two notions of exponential map for Lie groups and for Riemannian metrics do not coincide in general.

Due on Friday April 17