Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 7

1. Let G be a Lie group with a bi-invariant metric g.

(a) Show that the Levi-Civita connection of g satisfies

$$D_X Y = \frac{1}{2}[X, Y]$$

for left invariant vector fields X and Y.

(b) Show that the geodesics through the identity element are precisely the one-parameter subgroups of G (i.e. the two notions of exponential map for Lie groups and for Riemannian metrics coincide for bi-invariant metrics).

2. Let M be connected.

(a) Let $\phi, \psi \colon (M, g) \to (N, h)$ be isometries. Show that if ϕ, ψ and their differentials $d\phi$, $d\psi$ agree at a single point $p \in M$, then $\phi = \psi$.

(b) What upper bound does this suggest for the dimension of $Isom(M, g)$, assuming it is a manifold (it is!)?

3. A submanifold $N \subseteq (M, g)$ is called *totally geodesic* provided that whenever γ is a geodesic of M with $\gamma(0) \in N$, $\dot{\gamma}(0) \in T_{\gamma(0)}N$, then γ lies wholly in N.

Let H be a set of isometries of M, and define the fixed point set of H by

$$\mathrm{Fix}(H) := \{x \in M \mid \phi(x) = x \text{ for all } \phi \in H\}.$$

(a) Prove that each connected component of $\mathrm{Fix}(H)$ which is a submanifold of M is totally geodesic.

(b) Prove that $\mathrm{Fix}(H)$ is locally a submanifold of M, i.e each connected component is a submanifold of M.

(c) Must the dimension of $\mathrm{Fix}(H)$ be constant?

4. Let (M, g) be given. A smooth vector field X is called a *Killing field* if its local flow $\phi^t(x)$ is an isometry onto its image for all small t.

(a) Prove that X is a Killing field if and only if

$$\langle D_Y X, Z\rangle + \langle Y, D_Z X\rangle = 0$$

for all $Y, Z \in C^\infty(M)$. This says that the linear map $Y \mapsto D_Y X$ is antisymmetric with respect to the metric.

Hint: show first that $X \cdot \langle Y, Z\rangle = \langle L_X Y, Z\rangle + \langle Y, L_X Z\rangle$.

(b) Prove: if M is connected then X is determined everywhere by the values of $X(p)$ and $DX(p)$ at any given single point. This leads to an upper bound for the dimension of the space of Killing fields.

(c) Find all the Killing fields of flat $\mathbb{R}^3$.(Using b), we can verify that our list is complete.)

Due on Friday April 24