Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 8

1. (a) Show that the upper half-plane with the Lobatchevsky metric is complete.

(b) Show that the upper half-plane with the metric

$$g_{11}(x,y) = 1, \quad g_{22}(x,y) = \frac{1}{y}, \quad g_{12}(x,y) = 0$$

is not complete.

2. A connected Riemannian manifold M is called *non-extendible* if there is no connected Riemannian manifold N such that M is isometric to a proper open subset of N.

(a) Show that a complete Riemannian manifold is non-extendible.

(b) Give an example of an incomplete but non-extendible Riemannian manifold.

3. Let M be a Riemannian manifold, X a Killing field on M. Show that if M is complete then X is complete, i.e. the flow of X exists for all times.

4. *(Angle-excess Formula for Triangles in the Sphere.)* Let T be a geodesic triangle in S^2 with area A and angles α, β, γ.

(a) Show that $\alpha + \beta + \gamma = \pi + A$. *Hint:* First do the special case $\gamma = \pi/2$ by considering all 8 triangles subtended by the three great circles that bound T.

(b) Verify that the holonomy around the geodesic triangles is rotation by

$$\vartheta = \alpha + \beta + \gamma - \pi.$$

(c) It follows that the holonomy is given by the area enclosed: $\vartheta = A$.

(d) * Can you do something similar in the hyperbolic plane? *Hint:* Consider first an *ideal triangle*, that is, a triangle with all 3 of its vertices at infinity. You will have to find its area by doing an actual integral.

Due on Friday 1 May