Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 9

1. *(Length-Minimization)* Here is an alternate procedure for constructing minimizing geodesics in a complete Riemannian manifold M.

A curve $\gamma : [a, b] \to M$ is called *piecewise smooth* if it is continuous and there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that $\gamma| [t_i, t_{i+1}]$ is smooth. Fix $p, q \in M$ and let $\mathcal{C}$ be the set of all piecewise-smooth curves connecting p and q.

(a) A sequence $(\gamma_i)_{i\geqslant 1}$ in $\mathcal{C}$ such that $L(\gamma_i) \to \mathrm{d}(p, q)$ is called a *minimizing sequence.* Show there is $r > 0$, $k < \infty$, and a minimizing sequence $(\gamma_i)_{i\geqslant 1}$ such that each γ_i is a broken geodesic with at most k breaks and lies in $B_r(p)$.

(b) Show that a subsequence of $(\gamma_i)_{i\geqslant 1}$ converges to a length-minimizing broken geodesic γ in $\mathcal{C}$.

(c) Show that γ is smooth. Thus, γ is a minimizing geodesic connecting p and q.

2. *(Rays)* Let M be a complete, non-compact Riemannian manifold. A half-infinite geodesic $\gamma : [0, \infty) \to M$ is called a *ray* if it is length-minimizing (i.e. each finite segment is length-minimizing).

Show that for each $p \in M$ there is a ray emanating from p.

3. *(Two-Manifolds with a Killing Field)* Let M be a complete, connected, orientable Riemannian 2-manifold. Suppose M has a non-trivial Killing field X with at least one zero, say $X(p) = 0$. Prove that M is diffeomorphic to $\mathbb{R}^2$ or S^2.

The basic idea is to show that geodesic polar coordinates at p cover all of M except possibly one point. The implementation is a bit tricky.

(a) Write $D_r := B_r^{T_pM}(0)$. If $\exp_p |D_r$ is a diffeomorphism, show that X is given in geodesic coordinates $z = x + iy$ on the image of D_r by

$$X(z) = aiz, \qquad |z| < r,$$

for some $a \in \mathbb{R}$.

(b) Show that $\gamma := \exp_p(\partial D_r)$ is either a single point, or a smooth, embedded curve. In the latter case, $\exp_p |D_s$ is a diffeomorphism for some $s > r$. *Hint:* Examine the value of $|X|$ on γ. Recall that M is orientable. What might go wrong otherwise?

(c) Complete the proof.

4. *(Holonomy in Surfaces of Revolution)*

(a) Find the holonomy around a latitude circle γ in a surface of revolution. *Hint:* Let C be the cone tangent to the surface M along γ. Compare the holonomy around γ in C with that in M.

(b) Apply this to show that for a circle in S^2, we again have $\vartheta = A$, where A is the enclosed area (cf. Sheet 6, Exercise 1 and Sheet 8 Exercise 4).

Due on Friday 8 May