Differential Geometry II

D-MATH, FS 2015

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Exercise Sheet 3

1. Let $f : (M^m, g) \to (N^n, h)$ be a smooth map, with $m \geqslant n$. Define $|Jf(x)| := \det\big(df(x) \circ df(x)^T\big)$. The *coarea formula* states that

$$\int_M u(x)|Jf(x)|\, d\mu_g(x) = \int_N \int_{f^{-1}(y)} u(z)\, d\mu_{g^y}(z)\, d\mu_h(y)$$

where g^y is the induced metric on $f^{-1}(y)$. (Note that by Sard's theorem and the submersion theorem, y is a regular value of f, and $f^{-1}(y)$ is a smooth submanifold of M of dimension $m - n$ for μ_h-a.e. y in N, and we don't bother integrating over the measure-zero set of critical values. If you don't like this argument, only consider the case where f is a submersion.)

Now let f be a submersion. Decompose $T_pM = V_p \oplus H_p$, where $V_p := \ker(df_p)$ and $H_p := V_p^\perp$ are called the *vertical* and *horizontal* subspaces of T_pM with respect to f. We call f a *Riemannian submersion* if

$$df_p|H_p : H_p \to T_qN$$

is an isometry for all $p \in M$.

(a) Show that the Hopf fibration $f : S^3 \to S^2$ is a Riemannian submersion, if we adjust the radius of S^2 appropriately. (*Hint:* Use a well-chosen frame.) What is this magic radius?

(b) Use the co-area formula to compute $\mathrm{vol}(S^3)$.

2. Let $j : S^1 \to \mathbb{R}^2$ be the standard embedding. Let L be the twisted $\mathbb{R}$-bundle over S^1 (i.e. the Möbius strip). Let $\pi_i : \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the i-factor, $i = 1, 2$.

(a) Find an embedding $F : L \to \mathbb{R}^4$ such that $\pi_1 \circ F = j$ and $\pi_2 \circ F$ is linear on each fiber L_p, $p \in S^1$.

(b) Observe that F expresses L as a subbundle of the trivial bundle $S^1 \times \mathbb{R}^2 \to S^1$ of rank 2.

3. Let E_k be the k-twisted oriented $\mathbb{R}^2$-bundle over S^2. Prove: E_k and E_{-k} are isomorphic by an orientation-reversing bundle isomorphism over S^2.

4. Prove: TS^2 is isomorphic to E_2 as oriented $\mathbb{R}^2$-bundles over S^2. (Note that the standard orientation of S^2 induces an orientation on the fibers of TS^2).

5. The goal of this exercise is to construct Legendrian curves in S^3 (useful for Serie 2, Exercise 5). Let S^3 be the unit quaternions. Recall the left invariant vectorfields I, J, K on S^3 defined by

$$I(u) := ui, \quad J(u) := uj, \quad K(u) := uk, \qquad u \in S^3.$$

A curve γ in S^3 is called *Legendrian* if $\dot\gamma(t)$ is a linear combination of J and K for each t.

(a) Find a family of helix-like Legendrian curves as follows. Let $\operatorname{im}(c)$ be the image of the great circle $c(\theta) := e^{i\theta}$ in S^3. For each $0 < r < \pi/2$, define the torus

$$T_r := \{u \mid \operatorname{dist}_{S^3}(u, \operatorname{im}(c)) = r\}.$$

Observe that the vector field I is tangent to T_r. Now solve for the curves γ on T_r that are orthogonal to I at each point.

(b) For r very small, γ_r will lie very close to c, even though c has velocity vector I, yet γ_r is not allowed to go in the I-direction. Show that γ_r is much, much longer than c.

(c) Show that for any two points u, v in S^3 there exists a piecewise smooth Legendrian curve connecting u to v. (Use the Legendrian helices. Another way is note that $[J, K](p) = I(p)$, and look at the relations between the flows of the two vectorifields.)

Due on Friday March 20