Solution of Exercise sheet 10

1. The idea of the proof comes from a paper *The Existence of Complete Riemannian Metrics*, due by Katsumi Nomizu and Hideki Ozeki.

(a) Let (M, g) be a Riemannian metric. For any $p \in M$ we define $r(p)$ as the supremum of the positive numbers such that the closure of $B_{r(p)}(p)$ is compact. Note the that if $r(p) = \infty$ for some p, then M is already complete by Hopf-Rinov theorem. So we can assume that $r(p) < \infty$.
The proof goes as follows

(i) From the triangle inequality we get

$$|r(p) - r(q)| < d(p, q)$$

This implies that for any sequence p_i converging to p

$$\lim_{i\to\infty} |r(p_i) - r(p)| < \lim_{i\to\infty} d(p_i, p) = 0$$

and hence that the function $r : M \to \mathbb{R}$ is a positive continuous function.

(ii) A partition of unity argument show that there exist a smooth function $w : M \to \mathbb{R}$ such that $w(x) > \frac{1}{r(p)}$ for any $p \in M$.

(iii) We define $g' := w^2 g$. Since w is positive it is easy show that it is again a metric on M.

(iv) We denote with $B'_s(p)$ be a ball of radius s and center p in M with respect to the metric induces by g' and we denote with d' the distance function induces by g'. We prove that $B'_{\frac{1}{3}}(p) \subset B_{r(p)/2}(p)$. Let p, q be two points on M such that $d(p, q) \geqslant r(p)/2$. Let γ be a path connecting p with q parametrized by arc-length. Let $L(\gamma)$ be the length of γ with respect to d and let $L'(\gamma)$ be the length of γ with respect to d'. Then

$$\begin{aligned} L'(\gamma) &= \int_a^b w(\gamma(t)) \left(\langle\dot\gamma(t), \dot\gamma(t)\rangle_g\right)^{1/2} dt \\ &= w(\gamma(t_0))(b - a) = w(\gamma(t_0))L(\gamma) \end{aligned}$$

for any $t_0 \in [a, b]$ (here the second equality is just the mean values theorem for the integration). Now since

$$|r(\gamma(t_0)) - r(p)| < d(\gamma(t_0), p) \leqslant L(\gamma)$$

we get $r(\gamma(t_0)) < r(p) + L(\gamma)$. Then we conclude

$$L'(\gamma) > L(\gamma)/(r(p) + L(\gamma)).$$

Since $L(\gamma) \geqslant r(p)/2$ we get $L'(\gamma) > 1/3$ and so $d'(p,q) > 1/3$ as γ was arbitrary. Then we have proven

$$M/\overline{B_{r(p)/2}(p)} \subset M/\overline{B'_{\frac{1}{3}}(p)}$$

which implies

$$B'_{\frac{1}{3}}(p) \subset B_{r(p)/2}(p).$$

(v) With this, we can easily prove that (M, g') is complete. Namely, take $(p_k)_k$ a Cauchy sequence for d'. Then there is $N \in \mathbb{N}$ such that for every $i \geqslant N$, we have

$$d'(p_N, p_i) < \frac{1}{3}$$

Hence, $p_i \in \overline{B}'_{\frac{1}{3}}(p_N) \subset \overline{B}_{r(p_N)/2}(p_N)$ which is a compact set by definition of $r(p_N)$. Hence there is a subsequence of $(p_k)_k$ that converges, and so the Cauchy sequence converges. Therefore, (M, g') is complete.

(b) By above we can assume that (M, g) is a complete Riemannian manifold. Fix $p \in M$. Then the function $d(p, -) : M \to \mathbb{R}$ is a continuous function.

(i) A partition of unity argument show that there exist a smooth function $w(q)$ such that $w(q) > d(p,q)$.

(ii) Define $g' := e^{-2w}g$. Let γ be a length minimizing path between p and q with respect to g parametrize by arc length then $d(p, \gamma(t)) = t$ and

$$||\dot{\gamma}(t)||_{g'} = e^{-w(\gamma(t))} ||\dot{\gamma}(t)||_g = e^{-w(\gamma(t))}$$

(iii) The length of γ with respect to g' is

$$L'(\gamma) := \int_0^L ||\dot{\gamma}(t)||_{g'} = \int_0^L e^{-w(\gamma(t))} < \int_0^L e^{-t} < 1$$

then this is bounded.

(c) Come back to the point (ii) a). We need to define a new smooth function $\tilde{w}$ that doesn't creates troubles in the proof. More precisely since K is compact there exists a smooth function $\tilde{w}$ such that

$$\tilde{w}(q) := \begin{cases} 1 & \text{if} \quad q \in K \\ > \frac{1}{r(q)} & \text{if} \quad q \notin V \end{cases}$$

where V is some pre-compact open set containing K. Now since any compact metric space is complete we can use the function $\tilde{w}$ instead of w and obtain the same result. (You have to do some changes: do you see where? ;)). Analogously for the case b).

2. (a) one vertex, four edges.

(b) The point is that the gluing preserves the metric if and only if the sum of the angles S_r is equal to 2π. Now as r goes to one S_r converges to zero. On the other hand as r goes to zero S_r converges to 6π (Fix r and increase the radius of the disk indefinitely: you get that the arcs becomes segments and end up with the

Euclidean octagon). In particular since the sum is given by a smooth function depending on r we conclude that there exists an r' such that this gluing is possible.

(c) Its Euler characteristic is $1-4+1=-2$ thus we get a genus 2 Riemannian surfaces.

This process can be extend for genus $g \geqslant 2$ Riemannian surfaces as follows: replacing eight arcs in this construction by $4g$ arcs, then denote the sums of this angles by S_r^g. The above strategy remains true and it turns out that S_r^g goes to zero as r goes to 1, whereas S_r^g go to $(4g-2)\pi/2$. Thus the gluing is possible.
Also you can see that if $g=1$ then the sum of the angles is always smaller than 2π, i.e this method does not give you a hyperbolic torus.

3. (a) See b).

(b) The fact that it is a cone this is already prove in the class. To prove that this is again a smooth manifold we use the following strategy. We can consider $V=\mathbb{R}^q$, and $W=\mathbb{R}^p$, then C is given as the zero set of a family of polynomials. Then C is defined to be the zero set of $r \geqslant 0$ functions $f_1, \dots f_r$. Consider the map $f=(f_1,\dots,f_r)$, this is a smooth function

$$f : \mathbb{R}^{p+q} \to \mathbb{R}^r$$

and $C=f^{-1}(0,\dots 0)$. It is easy see that the only critical point of the map is precisely the corner of the cone. Then if we remove this point we get that C is a submanifold.

4. (a) Let $p \in M$. Assume that there are two isometries $\sigma_p, {\sigma'}_p$ that fix p and reverse each geodesics trough p. In particular the differential of the two maps at p is the same. Therefore the two isometries are the same by ExSheet 7.

(b) By what we prove in c), we know that M is complete. Take $p \in M$ and $r>0$ such that $B_r(p)$ is geodesically convex. Then choose $w \in B_{r/3}(p)$ and $q \in M$. Then by Hopf-Rinow, there is $\eta \in T_pM$ such that $\exp_p(\eta)=q$ and there is $\nu \in T_pM$ such that $\exp_p(\nu)=w$. We want to get an explicit expression for $\sigma_w(q)$ in terms of smooth functions. Namely, as σ_r is an isometry, we have

$$\sigma_r(q)=\sigma_r(\exp_p(\nu))=\exp_{\sigma_r(p)}(d\sigma_r(p)\nu).$$

Now, σ_r is the 'reflection' in r, so therefore $\sigma_r(p)=\exp_p(2\nu)$ by geodesic convexity. Furthermore, as σ_r is an isometry, it preserves parallel transport in the following sense:
If X is a parallel vector field along γ, then $(\sigma_r)_*X$ is a parallel vector field along $\sigma_r \circ \gamma$.
In our case, we can look at the parallel vector field X running along γ defined by $t \mapsto \exp_p(t\nu)$ such that $X(0)=\eta$. Under σ_r, we map γ to γ' defined by $t \mapsto \exp_p((2-t)\nu)$ and X is mapped to the parallel vector field X' along γ' which has $X'(1)=-X(1) \in T_rM$. Therefore

$$d\sigma_r(p)\nu = X'(0) = -\mathcal{P}_\gamma(\eta)(2).$$

(Here $\mathcal{P}$ stands for parallel transport. Draw a figure, if this formula doesn't seems clear to you after the above discussion. You could also simply look at the case of

$\mathbb{R}^n$ to clarify this massaging of formulae.) In total, we get

$$\sigma_r(q) = \exp_{\sigma_r(p)}(d\sigma_r(p)\nu) = \exp_{\exp_p(2\nu)}(-P_{t\to\exp_p(t\nu)}(\eta)(2)).$$

Thus smoothness of $\sigma : M \times M \to M$ follows from smooth dependence of solutions of ODE's on initial conditions.

(c) Let $\gamma : (a,b) \to M$ be a geodesic. Choose $\frac{a+b}{2} < t_0 < b$, put $p := \gamma(t_0)$. Then $\gamma' := \sigma_p(\gamma(t_0 - (t - t_0)))$ extend γ beyond b. Then by Hopf-Rinow M is complete.

(d) Note that

$$\tau_{\gamma,-} : \mathbb{R} \to Isom(M)$$

is well defined since $\tau_{\gamma,t}$ is the composition of isometries. On the other hand it is easy see

$$\tau_{\gamma,s}\gamma(t) = \sigma_{\gamma(s/2)}\sigma_{\gamma(0)}(\gamma(t)) = \sigma_{\gamma(s/2)}\gamma(-t) = \gamma(s+t).$$

and so $\tau_{\gamma,s} \circ \tau_{\gamma,t}(\gamma(0)) = \tau_{\gamma,s+t}(\gamma(0))$. Furthermore, if X is a parallel vector field along γ, then under $\sigma_{\gamma 0}$ it gets mapped to the parallel vector field X' along $\gamma' : t \mapsto \gamma(-t)$ having initial condition $X'(0) = -X(0)$ i.e.$X'(t) = -X(-t)$. Then applying $\sigma_{\gamma(s/2)}$ to X', we get a parallel vector field along $\gamma'' : t \mapsto \gamma(s+t)$ such that $X''(-s/2) = -X'(-s/2) = X(s/2)$, i.e. $X''(t) = X(t+s)$. Thus

$$d\tau_{\gamma,s}(\gamma(0))X(0) = X(s)$$

and thus

$$d\left(\tau_{\gamma,t+s}\right)_{\gamma(0)} = d\left(\tau_{\gamma,s} \circ \tau_{\gamma,t}\right)_{\gamma(0)}.$$

Hence the two isometries are the same, i.e

$$\tau_{\gamma,t+s} = \tau_{\gamma,s} \circ \tau_{\gamma,t}.$$

by ExSheet 7.

Remark. By Hopf-Rinow there exists a minimizing geodesic $\gamma : [a,b] \to M$ between any two pair $p, q \in M$. Let $p' := \gamma(\frac{b-a}{2})$. Then

$$\sigma_{p'}(p) = \sigma_{p'}(\gamma(0)) = q$$

and so $Isom(M)$ acts transitively on M.

5.

Remark (A different viewpoint for Symmetric Spaces)**.** We suggest to have a look of the following lecture: www.math.uni-augsburg.de/ eschenbu/symspace.pdf. They contain a lot of nice examples and proofs!

Below we summarize the consequences of exercise 4. We define a symmetric as a Riemannian manifold M equipped with a map $\sigma : M \times M \to M$ such that $\sigma_p(-) \in Isom(M)$ and

$$\sigma_p(p) = p, \quad d(\sigma_p)_p = -id$$

Now by point c) and d) we know that this implies that M is complete and homogeneous. Now fix a point $p \in M$ and define

$$K_p := \{g \in Isom(M) : gp = p\}$$

the isotropy group at p. It turns out that the differential of any $g \in K_p$ at p is an orthogonal transformation of T_pM but since any isometry f is completely determined by $f(p)$ and df_p we conclude that there is an inclusion as a closed subgroup

$$\theta : K_p \to O(T_pM)$$

defined by $\theta(g) := dg_p$. This in particular implies that K_p is compact. On the other hand assume that M' is a homogeneous Riemannian manifold equipped with an isometry $\sigma_p(-) \in Isom(M')$ such that

$$\sigma_p(p) = p, \quad d(\sigma_p)_p = -id \quad (*)$$

Then we define $\sigma_q := g^{-1}\sigma_p g$ where $g \in Isom(M')$, $g(p) = q$. Thus we conclude that a symmetric spaces can be defined as follows: *A symmetric spaces is a homogeneous Riemannian manifold M equipped with an isometry σ_p at some point $p \in M$ that satisfy (*).*

Moreover using some basic knowledge of algebra I (group action on a set) we may identify M as the set $Isom(M)/K_p$, i.e there is a bijection $IsomM/K_p \to M$ given by

$$gK_p \to gp$$

In particular it is possible to prove that $Isom(M)$(see for instance Helgason'book: *Differential Geometry, Lie Groups, and Symmetric Spaces*) has a smooth structure, that the above map is a diffeomorphism and that

$$\dim(M) = \dim(Isom(M)) - dim(K_p).$$

All of this data put together tells us that we can visualize a symmetric space as a quotient of groups.

(a) Consider the hyperbolic space with the *Hyperboloid model*

$$H^n := \left\{x \in \mathbb{R}^{n+1} : \langle x, x\rangle_L = -1\right\}$$

where the metric (called Lorentzian metric) is given by

$$\langle x, y\rangle_L := \left(\sum_{i=1}^{n} x_i y_i\right) - x_{n+1}y_{n+1}$$

In particular $(H^n, \langle -, -\rangle_L)$ is Riemannian manifold isometric to the Poincare n-disk model (check that by yourself ;) !). In this case the isometry is given by the "Lorentzian reflection" defined by

$$\sigma_x(y) := -y + 2\langle y, x\rangle_L\, x$$

(b) The set S_+ of symmetric real positive definite n-matrices can be visualized as an open subset of the vector space of symmetric real n-matrices. In particular the tangential spaces $T_P S_+$ is isomorphic to the set of symmetric real n-matrices. The metric on S_+ can be written via

$$\langle A, B\rangle_P := \operatorname{tr}\left(AP^{-1}BP^{-1}\right)$$

We define a smooth group action $GL(n, \mathbb{R}) \to S_+$ via $g(P) := gPg^T$. In particular it is easy see

$$dg_P(A) = gAg^T$$

for any $A \in S_+$. A simple calculation show that each g defines an isometry on $(S_+, \langle -, - \rangle)$. Moreover the action is transitive: from linear algebra any symmetric real positive definite n-matrix can be written as gg^T for some $g \in GL(n, \mathbb{R})$, i.e for any $P \in S_+$, $P = gg^T = g(Id)$. Consider the smooth map $\sigma_{Id} : S_+ \to S_+$ defined by $\sigma_{Id}(P) := P^{-1}$, note

$$\sigma_{Id}(Id) = Id, \quad d(\sigma_{Id})_P A = -P^{-1}AP^{-1}$$

for $A \in T_P S_+$. A straightforward calculation show that σ_{id} is an isometry. Moreover since $d(\sigma_{Id})_{Id} A = -A$ we conclude that $(S_+, \langle -, - \rangle)$ is a symmetric spaces.

(c) We don't have an answer (yet). ;) Helpful suggestions are welcome :)