Differential Geometry II

D-MATH, FS 2015

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Solutions of Exercise sheet 11

1. This was proven in the class or alternatively follows from unbundling definitions in local charts.

2. (a) Recall from exercise sheet 7 that for left invariant vectorfields X, Y we have

$$D_X Y = \frac{1}{2}[X, Y] \quad (*)$$

Moreover from differential geometry I we know that Lie bracket preserves left-invariant vectorfields.

$$\begin{aligned} R(X,Y)Z &= -\frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] + \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[[X,Y],Z] \end{aligned}$$

where the last equality uses the Jacobi identity of the Lie bracket. For the second equality we use the property of the Levi-Civita connection. We have

$$\mathrm{Rm}(X,Y,X,Y) = \left\langle \frac{1}{4}[[X,Y]X], Y \right\rangle$$

On the other hand we have

$$\begin{aligned} \mathrm{Rm}(X,Y,X,Y) &= \langle -D_X D_Y X, Y\rangle + \langle D_Y D_X X, Y\rangle + \langle D_{[X,Y]}X, Y\rangle \\ &= -Y\langle D_Y X, Y\rangle + ||D_Y X||^2 \\ &+ Y\langle D_X X, Y\rangle - \langle D_X X, D_Y Y\rangle \\ &+ [X,Y]\langle X,Y\rangle - \left\langle \frac{1}{2}[[X,Y],Y], Y \right\rangle \end{aligned}$$

Since the metric is bi invariant we have $Z\langle X,Y\rangle = 0$ for any left-invariant vector fields X, Y, then by (*) the above expression reduces to

$$||D_Y X||^2 + \left\langle \frac{1}{2}[[X,Y],Y], Y \right\rangle$$

But the second terms is equal to $-2\mathrm{Rm}(X,Y,Y,Y)$ and hence is zero.

(b) Let G be a Lie group and let $V, W \in P \subset T_gG$, then consider the two vectors $dL_{g^{-1}}V, dL_{g^{-1}}W \in T_{id}G$. In particular the two left invariant vector fields $V(-), W(-)$ satisfy $V(g) = V$ and $W(g) = W$.

Now by exercise sheet 1 we know that S^3 has a bi-invariant metric (the one induced by the inclusion in $\mathbb{R}^4$). Let V, W be an orthonormal basis of $P \subset T_pS^3$, let $V(-), W(-)$ the two left invariant vector fields satisfy $V(g) = V$ and $W(g) = W$. Write $V(id) = ai + bj + ck$, $W(id) = di + ej + fk$. Now since the metric is bi invariant we get that $V(id)$ $W(id)$ are again an orthonormal pair of vectors on $T_{id}S^3$

$$\begin{aligned}\mathrm{Rm}(V,W,V,W) &= \frac{1}{4}\,||[V(id),W(id)]||^2 \\ &= \frac{1}{4}\,||2\,(b_1c_1 - c_1b_2)\,i + 2\,(a_1c_2 - c_1a_2)\,j + 2\,(a_1b_2 - b_1a_2)\,k||^2\end{aligned}$$

where the last equality come from the explicit computation via the Lie bracket of S^3. Note that the results of the computation can be viewed as a cross product between $V(id)$ and $W(id)$, but since they are orthonormal we conclude

$$\mathrm{Rm}(V,W,V,W) = 1$$

(c) It follows from point a).

3. We do the proof in several steps.

(a) We use the geodesic normal coordinates x^1, x^2 near p on $B_R(p)$, i.e p is sent to 0 and let $e_1 = \frac{\partial}{\partial x^1}$, $e_2 = \frac{\partial}{\partial x^2}$ be the local frame near p. Now with respect to these coordinates we have that the metric g satisfies

$$g_{ij}(0) = \delta_{ij}, \quad \frac{\partial}{\partial x^k} g_{ij}(0) = 0, \text{ for } i,j,k = 1,2 \quad \Gamma^k_{ij}(0) = 0$$

this means that for any $z \in B_R(p)$

$$g_{ij}(z) = \delta_{ij} + O(|z|^2)$$

Therefore

$$K(p) = \mathrm{Rm}(e_1,e_2,e_1,e_2) = R_{1212}(0) = R^2_{121}(0)$$

(b) Using the above exercise 1, we have in geodesic normal coordinates

$$R^2_{121} = -\frac{\partial}{\partial x^2}\Gamma^2_{11} + \frac{\partial}{\partial x^1}\Gamma^2_{21}$$

Since the Chrystoffel symbols are given by

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$$

Thus a formal explicit computations (at 0) give us that

$$K(p) = -\frac{\partial}{\partial x^2}\Gamma^2_{11} + \frac{\partial}{\partial x^1}\Gamma^2_{21} = \frac{1}{2}\left(g_{22,11}(0) + g_{11,22}(0) - 2g_{12,12}(0)\right)$$

where $g_{ij,lk}$ means $\frac{\partial^2}{\partial x^k \partial x^l} g_{ij}$.

(c) On the other hand if we expand g_{ij} near 0 we get

$$g_{ij}(x) = \delta_{ij} + \frac{1}{2} g_{ij,11}(x_1)^2 + g_{ij,12}x_1x_2 + \frac{1}{2} g_{ij,22}(x_2)^2 + O(|x|^3)$$

Now fix an $r < R$, then

$$A(r) := \int_{|x|<r} \left(\det\left(g_{ij}(x)\right)\right)^{1/2} dx^1 dx^2$$

Thus if we insert the above expansion a straightforward calculation (expand also the square root and only keep terms up to order 2) gives

$$A(r) = \pi r^2 + \frac{\pi}{4} Z r^4 + O(r^5)$$

where $Z = -\frac{1}{4}\left(g_{11,11}(0) + g_{11,22}(0) + g_{22,11}(0) + g_{22,22}(0)\right)$.

(d) Now we want to show that $K(p) = \lambda Z$ for some number λ. Let $X = \left(X^1, X^2\right)$ be a tangent vector at 0. Denote with X the constant vectorfield $X(x) := X^i e^i$. Recall that near 0 the path

$$\gamma(t) := \left(tX^1, tX^2\right)$$

is a geodesic. Note also that $\dot\gamma(t) = X$. Then we get

$$0 = D_\gamma \gamma = \frac{\partial^2}{\partial t^2}\gamma^k + \dot\gamma^i(t)\dot\gamma^j(t)\Gamma^k_{ij}(\gamma(t)) = \dot\gamma^i(t)\dot\gamma^j(t)\Gamma^k_{ij}(\gamma(t)) = X^i X^j \Gamma^k_{ij}(\gamma(t))$$

Now if we differentiate the above equation at $t = 0$ we get a relation between the first derivative of the Chrystoffel's symbols. More precisely they are

$$g_{11,11}(0) = 0 = g_{22,22}(0), \quad g_{11,22}(0) = g_{22,11}(0) = -2g_{12,12}(0),$$

If we insert these relation in $K(p)$ and in Z we get

$$K(p) = \frac{3}{2} g_{11,22}(0), \quad Z = -\frac{1}{2} g_{11,22}(0)$$

(e) We can now write

$$A(r) := \pi r^2 - \frac{\pi K(p)}{12} r^4 + O(r^5)$$

Now since $A(r) = \int_0^r C(s)ds$ we get that $C(r) = \frac{d}{dr}A(r)$. This give us the formula for $C(r)$.

4. Fix a $p \in M$. By applying a translation on $\mathbb{R}^3$ (isometry), we may assume that $p = 0$ and by a rotation (isometry), we may assume further, that the tangent plane T_pM equals to $\{z = 0\}$. Now look at $\phi : M \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$ and for $0 \in M$, we have that $d\phi(0, 0)$ is an isomorphism. Therefore by inverse function theorem, there is a neighborhood in $B_r(0) \subset \mathbb{R}^2$ such that $\phi^{-1} : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a diffeomorphism onto its image. Hence over $B_r(0)$ there is a function $u : B_r(0) \to \mathbb{R}$ such that

$$f(x, y) := (x, y, u(x, y)) \in M$$

for all $(x, y) \in B_r(0, 0)$ and so we established that locally M is a graph over the first two coordinates.
As a consequence, $u(0, 0) = 0$, $\partial_x u(0, 0) = \partial_y u(0, 0) = 0$ and a local frame is given by

$$\partial_x f(x, y) = (1, 0, u_x(x, y)), \quad \partial_y f(x, y) = (0, 1, u_y(x, y))$$

With respect to this frame we get

$$g_{11} = 1 + (\partial_1 u)^2 \qquad g_{12} = \partial_1 u \partial_2 u \qquad g_{22} = 1 + (\partial_2 f)^2$$

and therefore

$$\begin{array}{ll} \partial_1 g_{11} = 2u_x u_{xx} & \partial_2 g_{11} = 2u_x u_{xy} \\ \partial_1 g_{12} = u_{xx} x u_y + u_x x u_{xy} & \partial_2 g_{12} = u_{xy} x u_y + u_x x u_{yy} \\ \partial_1 g_{22} = 2u_y u_{yy} & \partial_2 g_{22} = 2u_y u_{xy} \end{array}$$

where $u_x = \partial_1 u$ and $u_y = \partial_2 u$. Therefore, $\Gamma_{ij}^k(0,0) = 0$. Hence,

$$K(p) = \frac{1}{2}(\partial_1\partial_1 g_{12}(0,0) - \partial_1\partial_1 g_{22}(0,0)) = u_{xx}(0,0)u_{yy}(0,0) - u_{xy}^2(0,0)$$

which the determinant of the Hessian of u at $(0,0)$. Since this matrix is diagonalizable, the determinant is equal to the products of the two principal curvatures.

5. (a) Recall form exercise 4 d) of the previous exercise sheet that each geodesic γ on M defines a one parameter subgroup (a group homomorphism) $\tau_{\gamma,(-)} : \mathbb{R} \to Isom(M)$, via

$$\tau_{\gamma,t} = \sigma_{\gamma(t/2)} \circ \sigma_{\gamma(0)}.$$

Let X be a parallel vector field along γ. We already proved that $\tau_{\gamma,t}$ is the isometry that sends $X(s)$ to $X(t+s)$, but since X is parallel along γ we conclude that $d\tau_{\gamma,t}$ acts as the parallel transport along γ. $\tau_{\gamma,t}$ is called transvections along γ. Now let $p \in M$ $V \in T_pM$ and let γ be the geodesic with initial velocity V. Now we can consider $\tau_{\gamma,t}$ as the flow of a vector field: let $\phi^- : M \times \mathbb{R} \to M$ defined by

$$\phi^t(q) = \tau_{\gamma,t}(q)$$

then let X be the vectorfield defined by

$$X(q) = \frac{\partial}{\partial t}\phi^t(q)$$

then X is a Killing fields with $X(p) = V$. Now let $W \in T_pM$, let $\delta(s)$ be a path starting at p with initial velocity W given by $\delta(s) = \exp_p(sW)$. We have

$$\begin{aligned} D_W X(p) &= D_{\dot\delta(s)} X(p)\Big|_{s=0} \\ &= \frac{D}{ds}\left(\frac{\partial}{\partial t}\phi^t(\delta(s))\right)\Bigg|_{t=s=0} \\ &= \frac{D}{dt}\frac{\partial}{\partial s}\phi^t(\delta(s))\Bigg|_{s=t=0} \end{aligned}$$

where the last equation comes from the torsion free property of D. Since

$$\frac{\partial}{\partial s}\Bigg|_{s=0} \phi^t(\delta(s)) = d\tau_{\gamma,t} W$$

and $d\tau_{\gamma,t}$ acts like the parallel transport along γ we conclude that the above expression is 0.

(b) Let X, Y, Z be 3 Killing fields with $DX(p) = DY(p) = DZ(p) = 0$, then we have

$$R(X,Y)Z(p) = [[X,Y],Z](p)$$