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Solutions of Exercise sheet 12

1. This is a linear algebra exercise. We can find the dimension as follows

(a) the first non trivial situation is when R_{ijkl} looks like R_{ijij}, in this case the dimension is $\binom{n}{2}$,

(b) the second case is when three indices are distinct and one is repeated, in this case the dimension is $3\binom{n}{3}$,

(c) the last case is when the four indices are distinc, in this case the dimesnion (by Bianchi's identity) is $2\binom{n}{4}$.

Hence the sum is given by $\frac{1}{12}n^2\left(n^2-1\right)$.

2. (a) $\frac{1}{12}n^2\left(n^2-1\right)=1$ for $n=1$.

(b) Let $A \in O(T_pM)$, let e_1, e_2 be an orthonormal basis of T_pM and let $K(p)$ be the sectional curvature at p. Then by multi-linearity

$$Rm\left(A\left(e_1\right), A\left(e_2\right), A\left(e_1\right), A\left(e_2\right)\right)=(\det(A))^2 K(p)$$

(c) By definition and b), for some orthonormal basis,

$$R_{1212}=K(p)$$

Using a), we get the result.

3. Since the hyperbolic space is a homogeneous space (as it is symmetric) and since all these tensors are isometry-invariant, we conclude that our result is independent of the point $p \in H^n$. On the other hand to compute the Riemann curvature tensor, we need to know our metric (in a Taylor expansion) only up to order 2 (Rm depends only on Christoffel symbols and their first derivatives.). Consider the disk model then its metric g can be approximated (via a Taylor expansion) at $p=(0,\ldots,0)$ by

$$g_{ij}=4\delta_{ij}-8x^ix^j+o(||x||^3).$$

Using this trick is it now easy to calculate the desired tensor at $(0,\ldots,0)$. To check whether your calculations were correct, you have to find that all sectional curvatures are equal to -1.

4. We start with some definitions first. For a vector field X on G we define ad_X : $C^\infty(TM) \to C^\infty(TM)$ via

$$ad_X(Y) := [X, Y]$$

In particular we are going to consider its restriction $ad_X : T_eG \to T_eG$. On the other hand we denote with $AD_a : G \to G$ the group automorphism sending $g \to aga^{-1}$ and with $Ad(a) : T_eG \to T_eG$, the map

$$Ad(a)(X) = d\left(AD_a\right)_e(X).$$

(see exercise sheet 1, exercise 5).

(a) B being bilinear follows from linearity of the objects in the definition. Next, we prove that B is Ad-invariant, i.e $B(Ad(a)X, Ad(a)Y) = B(X,Y)$. First we prove that

$$Ad_a \circ ad_X \circ (Ad_a)^{-1} = ad_{Ad_aX}$$

for any $a \in G$, $X \in T_eG$. Indeed

$$\begin{aligned} Ad_a \circ ad_X \circ (Ad_a)^{-1} y &= Ad_a\left[X, (Ad_a)^{-1} Y\right] \\ &= \left[Ad_aX, Ad_a\,(Ad_a)^{-1} Y\right] \\ &= [Ad_aX, Y] \\ &= ad_{Ad_aX}Y. \end{aligned}$$

Then

$$\begin{aligned} B(Ad(a)X, Ad(a)Y) &= \operatorname{tr}\left(ad_{Ad_aX} \circ ad_{Ad_aY}\right) \\ &= \operatorname{tr}\left(Ad_a \circ ad_X \circ (Ad_a)^{-1} \circ Ad_a \circ ad_Y \circ (Ad_a)^{-1}\right) \\ &= \operatorname{tr}\left(ad_X \circ ad_Y\right) \\ &= B(X,Y). \end{aligned}$$

since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. (This last identity on the trace also tells you that B is symmetric.)

(b) Recall that any compact Lie group G carries a Haar measure μ with $\mu(G) = 1$. Now choose a basis on T_eG and let $\langle -, - \rangle$ be the inner product on T_eG that makes the above basis orthonormal. In particular, μ can be used to put an Ad invariant inner product on T_eG via

$$h(X,Y) := \int_G \langle Ad_aX, Ad_aY \rangle \, d\mu(a).$$

If you extend this h as a left-invariant metric to G, then this extended metric will also be right-invariant. (This was already shown in the exercise 5, exercise sheet 1). Now consider $O(T_eG) \subset GL(T_eG)$ with respect to the inner product h.

(i) Consider the Lie group homomorphism $Ad_{(-)} : G \to GL(T_eG)$. Since h is Ad invariant we conclude that $Ad_a \in O(T_eG)$ for any $a \in G$.

(ii) The differential of $Ad_{(-)}$ at the identity $e \in G$

$$d\left(Ad_{(-)}\right)_e : T_eG \to T_{Id}GL(T_eG)$$

is given by $d\left(Ad_{(-)}\right)_e(X) = ad_X$. Note that $T_{Id}GL(T_eG) \cong \mathbb{R}^{n^2}$.

(iii) By point (i) we have that ad_X is contained in $T_{Id}O(T_eG)$. Now let γ be a path on $O(T_eG)$ starting at the identity. Let $A = \dot{\gamma}(0)$. Then for any $v, w \in T_eG$

$$h(\dot{\gamma}(t)v, \dot{\gamma}(t)w) = h(v,w)$$

for any t. Thus

$$0 = \frac{d}{dt}|_0 h(\dot\gamma(t), \dot\gamma(t)) = h(Av, w) + h(v, Aw).$$

Hence, $T_{Id}O(T_eG)$ may be identified with the space of matrices A that satisfy $h(Av, w) + h(v, Aw) = 0$.

(iv) If we take the complexification $(T_eG \otimes \mathbb{C}, h_{\mathbb{C}})$ of the inner product space (T_eG, h) (i.e we extend h to an Hermitian inner product $h_{\mathbb{C}}$) we get that $A \in T_{Id}O(T_eG)$ can be extended as a Hermitian matrix on $(T_eG \otimes \mathbb{C}, h_C)$. Now by the spectral theorem A is diagonalizable. Let λ be an eigenvalue of A, then

$$\begin{aligned} \lambda h_{\mathbb{C}}(z, z) &= h_{\mathbb{C}}(Az, z) \\ &= -h_{\mathbb{C}}(z, Az) \\ &= -\bar\lambda h_{\mathbb{C}}(z, z) \end{aligned}$$

i.e its eigenvalues are purely imaginary.

(v) Since ad_X is contained in $T_{Id}O(T_eG)$, we can consider it as a Hermitian matrix with purely imaginary eigenvalues. Therefore,

$$B(X, X) = \operatorname{tr}(ad_X \circ ad_X) = \operatorname{tr}\left(A^2\right) = \sum \lambda^2 \leqslant 0.$$

(c) By the above, $g = -B$ is a bi-invariant metric on T_eG in case B is non-degenerate. This is exactly the case if G is semi-simple. (If you don't know this, simply take this as a definition or look it up in a Lie algebra textbook ;)) Then give yourself an orthonormal left invariant basis e_i and some left-invariant vector fields X, Y. As the left-invariant vector fields e_i form a global frame, it is enough to prove the identity for left-invariant vector fields as Rc and g are tensors. Using exercise 2a) of exercise sheet 11 we have

$$\begin{aligned} Rc(X, Y) &= \sum_{i=1}^{n} Rm(X, e_i, Y, e_i) \\ &= \sum_{i=1}^{n} g\left(R(X, e_i)Y, e_i\right) \\ &= \frac{1}{4}\sum_{i=1}^{n} g\left(\left[\left[X, e_i\right] Y\right], e_i\right) \\ &= -\frac{1}{4}B(X, Y) = \frac{1}{4}g(X, Y) \end{aligned}$$

(d) We need that the Lie algebra is semi-simple, i.e it is the direct sum of simple Lie algebras. Do you know some examples/counter examples? ;)

Happy Holidays!