Differential Geometry II

D-MATH, FS 2015

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Solution of Exercise sheet 8

1. (a) We already know that the hyperbolic upper half plane $\mathbb{H}$ is geodesically complete. Then this is a consequence of the Hopf-Rinow theorem.

(b) This space is not geodesically complete. In this case, we can easily compute the Christoffel symbols. Namely the only non-zero entry is $\Gamma^2_{22}(x,y) = -\frac{1}{2}\frac{1}{y}$
This gives us the geodesic equation for $\gamma : \mathbb{R} \to \mathbb{R} \times \mathbb{R}_+$

$$\ddot{\gamma}_1 = 0 \qquad \ddot{\gamma}_2 = \frac{1}{2\gamma_2}(\dot{\gamma}_2)^2$$

The first equation gives $\gamma_1(t) = at + b$. Requiring that $\gamma_1(0) = 0$ and $\dot{\gamma}_1(0) = 0$, we get $\gamma_1 = 0$.
Thus we now have to solve the second equation with $\dot{\gamma}_2(0) = -1$ and $\gamma_2 = 1$, for which we know that its speed $\frac{(\dot{\gamma}_2)^2}{\gamma_2}(t)$ is constant and thus equal to 1. (This is actually equivalent to our second equation.) Hence it is not difficult to solve this new equation (by separation) to get the unique solution

$$\gamma_2(t) = (\frac{t}{2} - 1)^2.$$

But we see that this expression remains only non-zero for $t < 2$ and we cannot continue this geodesic any further. So this space is not geodesically complete.

2. Let γ_p be the integral curve at $p \in M$ of X. Assume that it is defined on a maximal interval I and by short term existence of ODE, we know that I is open. So $I = (a,b)$ for $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Since it is the integral curve of a Killing field we get that $\langle \dot{\gamma}_p, \dot{\gamma}_p \rangle$ is constant on I. Assume that $b < \infty$.

$$\int_0^b \langle \dot{\gamma}_p, \dot{\gamma}_p \rangle = b \langle \dot{\gamma}_p(a), \dot{\gamma}_p(a) \rangle .$$

Therefore, γ_p can be extended on $[a,b]$ via

$$\gamma_p(b) := \lim_{t \to b} \gamma_p(t),$$

in the following way: Let t_i be a sequence in I that converges to $b \in \mathbb{R}$. So t_i is a Cauchy sequence in I. Now consider the sequence $\gamma_p(t_i)$ we get

$$\operatorname{dist}(\gamma_p(t_i), \gamma_p(t_j)) \leqslant \langle \dot{\gamma}_p(a), \dot{\gamma}_p(a) \rangle |t_i - t_j|$$

whence $\gamma_p(t_i)$ is a Cauchy sequence and one checks that $\gamma_p(b)$ is well defined. But then $b \in I$ and we can extend γ_p to $I \cup (b, b+\epsilon)$ which is a contradiction to the maximality of I.

3. Assume that M is extendible and complete, then M can be considered as an open set of a bigger manifold N. In particular M is open in the topology of N. Now let p be a point of the boundary of M in N. Let $i(p)$ be the injectivity radius at p and let $B_{i(p)}(p)$ be the open ball of radius $i(p)$ centered at p. Then $\exp(B_{\frac{i(p)}{n}}(p))$ is open and $\exp(B_{\frac{i(p)}{n}}(p)) \cap M$ is open and not empty for $n \in \mathbb{N}$. Take $x_n \in \exp(B_{\frac{i(p)}{n}}(p)) \cap M$ and by construction $x_n \to p$. Since it is convergent it is also a Cauchy sequence, and so by completeness of M, we have $p \in M$. This is a contradiction an so $M = N$.

4. Consider $\mathbb{R}^2 \backslash \{(0,0)\} = \mathbb{C} \backslash \{(0)\}$ with its standard metric δ. Then its *universal cover* is given by $\mathbb{C}$ together with the covering map

$$\exp(-) \ : \ \mathbb{C} \to \mathbb{C}/\{(0)\}$$

where $\exp(z) := e^z$ is the usual complex exponential map (no geodesics here ;)). Denote with δ' the pullback of δ with respect to $\exp(-)$. Then $\exp(-)$ its a local isometry between $(\mathbb{C}, \delta')$ and $(\mathbb{C}\backslash\{(0)\}, \delta)$. Now take an open small circle C in $\mathbb{C}/\backslash\{(0)\}$ which is tangent to 0. Its preimage is a disjoint union of connected open sets $A_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. Let x_i be a Cauchy sequence in C that converges to 0 (with respect to δ). Since $\exp(-)$ is a local isometry it preserves Cauchy sequences and so the preimage of x_i in A_n for a fixed n is another Cauchy sequence y_i. Assume that y_i converges to $q \in A_n$. Since $\exp(-)$ induces an isometry

$$\exp(-)|_{A_n} \ : \ A_n \to \exp(A_n)$$

we conclude that $\exp(q) = 0$ which is a contradiction. Thus y_i is a Cauchy sequence that doesn't converge in $\mathbb{C}$ with respect to δ'.

We now show that $(\mathbb{C}, \delta')$ is non extendable. First note that the above covering map can be written using real coordinates as $f \ : \ \mathbb{R}^2 \to \mathbb{R}^2/\{(0,0)\}$

$$f(x,y) := (e^x \cos y, e^x \sin y)$$

and it is easy conclude that δ' with respect to the real coordinates is given by

$$e^x \cdot \langle -, - \rangle_{\mathbb{R}^2} \,.$$

This metric has the following property: for any point $p \in \mathbb{R}^2$ there exist exactly one geodesic γ through p which cannot be extended on all $\mathbb{R}$. (All the other geodesics are can be extended from $\mathbb{R}$ to $\mathbb{R}^2$). This fact can be proved a s follows: As locally $\exp(-)$ is an isometry, we know that locally geodecs on $(\mathbb{C}, \delta')$ are the inverse image of geodesics on $(\mathbb{C}\backslash\{(0,0)\}, \delta)$, which are simply lines. Hence geodesics on $(\mathbb{C}, \delta')$ are simply lifts of lines of $(\mathbb{C}\backslash\{(0,0)\}, \delta)$ and there is exactly one incomplete direction that hits $(0,0)$. Alternatively, you can write down the geodesic equation and solve it.

Now assume that $\mathbb{R}^2$ is extendable and let M be a 2 dimensional Riemannian manifold that contains $\mathbb{R}^2$. Let p be a point in the boundary of $\mathbb{R}^2$ in M. And let W be a geodesically convex open neighborhood of p in M. Let q be a point of $W \cap \mathbb{R}^2$, let $\tilde{\gamma}$ be the geodesic connecting q with p. From the uniqueness above it turns out that $\tilde{\gamma}$ initially coincide with the geodesic γ through q which cannot be extended to all of $\mathbb{R}$ (otherwise γ lies always in $\mathbb{R}^2$). Hence all the points on γ are in $\mathbb{R}^2$.

Now assume that you want to connect q with another point x of W not lying on $\tilde{\gamma}$, it turns out that there is a geodesic γ' connecting q with x and it is different from $\tilde{\gamma}$. Thus by the discussion above this geodesic is completely contained in $\mathbb{R}^2$, and so $x \in \mathbb{R}^2$. Varying $q \in W \cap \mathbb{R}^2$ gives you by the same argument that $W - p \subset \mathbb{R}^2$.

Hence the boundary is discrete and hence $\mathbb{R}^2 = M \backslash \{p_i : i \in I\}$. By Jordan's Curve theorem for $\mathbb{R}^2$, we know that any loop in $\mathbb{R}^2$ bounds a region diffeomorphic to the disc and one unbounded component. Thus by running around one of the punctures and applying this result, we see that there cannot be more than one puncture and that M is diffeomorphic to the sphere S^2 (but not necessarily isometric to the standard round sphere) and call the puncture $\{p\}$. This is a compact manifold and hence it has to have finite diameter $\sup\{d(x,y) : x, y \in M\}$, whereas on $(\mathbb{R}^2, \delta')$ the diameter is infinite. (Lifts of radial lines have infinite length.) This is a contradiction and therefore this example is not extendible.

5. (a) For any angle α we denote with L_α its "lune", i.e take two great circles starting at the north pole and assume that the angle between is α, then consider the two symmetric surfaces delimited, they are called lune and antipodal lune.
Let T be a triangle with angle α, β, γ, then there is a geometrical relation

$$A(L_\alpha) + A(L_\beta) + A(L_\gamma) = 4A(T) + A(S^2)$$

where $A(-)$ means area of $(-)$. Since $A(L_\alpha) = 4\alpha$, $A(S^2) = 4\pi$ the result follows.

(b) This can be verify geometrically. Draw the triangle T such that one of the edges lies on the equator the it is easy show that the holonomy is precisely $\alpha + \beta + \gamma - \pi$.

(c) This follows from a) and b).

(d) Since the geometry is hyperbolic here (curvature is -1) we expected an area defect formula, i.e.

$$\alpha + \beta + \gamma = \pi - A$$

Consider the Poincare disk model. An ideal triangle is a triangle centered in the center of the disk such that the edges are tangents at the vertices. It turns out that the angles are all equal zero. Then the above formula is true if the area of the ideal triangle is π. This area can be calculates using integration.
The general case can be similarly proven using integration.