Solution of Exercise Sheet 9

1. (a) Recall that the distance between two points p, q is given by

$$d(p,q) = \inf \{L(\gamma) \,:\, \gamma \text{ is a path connecting } p, q\}$$

This exercise can be thought of being an alternative proof of Hopf-Rinov theorem. Due to geodesical completeness, $\overline{B}_r(p)$ is compact for every $r \in \mathbb{R}_+$ and $p \in M$ as $\overline{B}_r(p)$ is a closed set contained in $\exp_p(\overline{B}_{r+\epsilon})$ which is compact.
Now given $p, q \in M$, then let $d := d(p,q) < \infty$ and as $\overline{B}_d(p)$ is compact, there is an $\epsilon > 0$ such that $B_\epsilon(x)$ is a geodesically convex neighborhood (i.e. any two points in this ball are connected by a unique geodesic and the dependence on the endpoints is smooth. Tho see that this exists simply consider $id \times \exp : TM \to M \times M : (p, v) \mapsto (p, \exp_p(v))$ and apply Inverse function theorem.) Now take $R = d + \epsilon$. For any $\frac{1}{n} < \frac{\epsilon}{3}$, we get by definition of the infimum, that there is a curve connecting $\gamma : [0,1] \to M$ connecting p and q and such that $l(\gamma) \leqslant d + \frac{1}{n}$. This curve has to lie in $B_R(p)$ for length reasons. Thus choose increasing times t_i^n for $i \in \{1, \ldots, K_n\}$ such that

$$d(\gamma(t_i^n), \gamma(t_{i+1}^n)) < \frac{\epsilon}{3}, \quad t_0^n = 0, \quad t_{K_n}^n = 1 \quad \text{ and } K_n = \left\lceil \frac{d + \frac{1}{n}}{\frac{\epsilon}{3}} \right\rceil - 1$$

Now we can connect the points $q_i := \gamma(t_i)$ with $q_{i+1} := \gamma(t_{i+1})$ with the unique geodesic ν_i^n in the geodesic neighborhood $B_\epsilon(q_i)$. These geodesic segments cannot leave $B_R(p)$ by triangular inequality. Now the concatenation μ^n of all of these K_n geodesics ν_i^n is a piecewise smooth geodesic and its length is $\leqslant K_n \frac{\epsilon}{3} \leqslant d + \frac{1}{n}$. Now the sequence $\{\mu^n\}_{n \geqslant N}$ is a minimizing sequence of piecewise smooth geodesics with $K_n \leqslant K_N$ breaks.

(b) With the same notation as in a), we fix some more points on the curve μ_n so that every curve has exactly K_N breaks. As $q_1^n \in \overline{B}_R(p)$ lie in a compact set, there is a convergent subsequence $q_1^{(n,1)}$ converging to q_1^∞. Now repeat this process, to extract successively subsequences $q_i^{(n,i)}$ such that $q_j^{(n,i)} \to q_j^\infty$ for $j \leqslant i$. Finally, you end up with convergent sequences $q_i^{(n,K_n)}$ to q_i^∞. We have $d(q_i^\infty, q_{i+1}^\infty) \leqslant \frac{\epsilon}{3}$ and so we can get the concatenated piecewise smooth geodesic μ^∞ made up of the geodesics ν_i^∞ as before. As the dependence on the endpoints is smooth in the geodesically convex neighborhood, the curves $\nu_i^{(n,K_n)}$ converge uniformly together with all their derivatives to ν_i^∞. Therefore, the length of μ^∞ has to converge to d by construction as length only depends on the curve and its first derivative.

(c) If γ is length minimizing, then for any $x \in \gamma$ we have that in $B_\epsilon(x)$ the restriction of γ is the unique length minimizing geodesic connecting different point of γ in $B_\epsilon(x)$. It follows that γ is smooth in $B_\epsilon(x)$.

2. Since M is complete, then the exponential map

$$\exp_p : T_pM \to M$$

is surjective. Then $M = \bigcup_{n\in\mathbb{N}} B_n(p)$. Since M is not compact, there must be a sequence p_n in M such that $d(p,p_n) = n$ and p_n is a a point on the boundary of $B_n(p)$. Let γ_n be a lenght minimizing geodesic connecting p with p_n. It can be written as

$$\gamma_n := \exp_p(nX_n)$$

for some unit tangent vector X_n. Since the unit ball in T_pM is compact, the sequence X_n has a convergent subsequence $\left\{X_{n_j}\right\}_j$. Let X be the limit of this subsequence. We show that $\gamma(t) := \exp_p(tX)$ is a ray. Assume that there exist an t_0 such that

$$d(p,\gamma(t_0)) = t_0 - \epsilon$$

for some $\epsilon > 0$. We need the following two facts:

(a) Since the exponential map is continuous in both the variables, there exists a $\delta > 0$ such that

$$d\left(\exp_p(t_0X), \exp_p(t_0Y)\right) < \epsilon$$

for any Y with $||X - Y|| < \delta$.

(b) Since X is a limit of a convergent subsequence there exists a $l > t_0$ such that

$$||X - X_l|| < \delta$$

Now by triangular inequality

$$\begin{aligned} d\left(p, \exp(lX_l)\right) &\leqslant d\left(p,\gamma(t_0)\right) + d\left(\gamma(t_0), \exp(lX_l)\right) \\ &\leqslant d\left(p,\gamma(t_0)\right) + d\left(\gamma(t_0), \exp(t_0X_l)\right) + d\left(\exp(t_0X_l), \exp(lX_l)\right) \\ &< t_0 - \epsilon + \epsilon + (l - t_0) \end{aligned}$$

where the last term comes from the fact that $\exp(tX_l)$ is the length minimizing geodesic connecting p with p_l. We conclude

$$d\left(p, \exp(lX_l)\right) < l$$

which is a contradiction.

3. (a) We first prove a general fact: Let M be a Riemannian manifold equipped with a Killing field. Assume that $X(p) = 0$ at some p. Then X is tangent to the geodesic spheres around p for small radii.

Let $r > 0$ small enough such that $\exp_p(-) : B_r(0) \subset T_pM \to M$ is a diffeomorphism onto its image. Now let $q \in \exp_p\left(B_r(p)\right)$, let ϕ^t be the flow of X. Since $X(p) = 0$ we have $\phi^t(p) = p$ for all $t \in \mathbb{R}$. Then from

$$d(p,q) = d(\phi^t(p), \phi^t(q)) = d(p, \phi^t(q))$$

it follow that the integral line trough $q \in B_r(p)$ lies on the geodesic sphere of radius s where $0 < s := d(p,q) < r$.

Assume from now on that M is a two dimensional Riemannian manifold. We want to prove that zeroes of non-trivial Killing fields are isolated in two dimensions. Indeed, consider the set $H = \{\phi^t : M \to M : t \in \mathbb{R}\}$ of isometries (we already proved in ExSheet 8, that Killing fields have complete flows.) and its fixed point set $N := Fix(H) = \{p \in M : X(p) = 0\}$ which we proved to be a manifold in ExSheet 7 with tangent space at $p \in N$ equal to the space of vectors fixed by $d\phi_t(p)$ for all $t \in \mathbb{R}$. But $d\phi_t(p) \in SO(2)$, which means that either there is a two dimensional space of fixed vectors ($d\phi_t(p) = id$) or $T_pN = \{0\}$. As X is non-trivial, we can't have the first option according to ExSheet 7 and so N is 0-dimensional, i.e. consists of isolated points.

From ExSheet 7, we already know that the length $\langle X, X\rangle$ of a Killing field X is constant along each integral line. The geodesic normal coordinates around p are given by a map

$$\exp_p : D_r \to M$$

where D_r is the open disk in $\mathbb{C} = \mathbb{R}^2$ of radius r. Then the vector fields on D_r that are tangent to all the circles C_s, $s < r$ and with constant length are of the form

$$X(z) = a(|z|)iz.$$

for some smooth function $a(|\cdot|) : [0, r^2] \to \mathbb{R}$.
Note that the integral curves of X with respect to these coordinates are the circles $\gamma_s : S^1 \to \mathbb{C}$

$$\gamma_s(t) := se^{ia(s)t}$$

for any $0 < s < r$.

(b) Let r sufficently small such that $\exp_p : D_r \to M$ is a diffeomorphism. We show that $\exp_p(\partial D_r)$ is the set $\partial B_r(p)$ of points with distance r from p.
Since M is complete we have that $\exp_p(\overline{D}_r)$ is compact and therefore closed as $\overline{B}_{r+\epsilon}(p)$ is compact. Therefore as $\overline{B}_r(p)$ is the smallest closed set containing $B_r(p)$ and $\exp_p(D_r) = B_r(p)$, we have $\partial B_r(p) \subset \exp_p(\partial D_r)$.
For the converse, for $v \in \partial D_r$ take a sequence $v_i \in D_r$ converging to v and so $\exp_p(v_i) \in B_r(p)$ converges to $\exp_p(v)$ by continuity, therefore $\exp_p(\partial D_r) \subset \partial B_r(p)$.

Note also that the proof of a) carry over this situation, and X is tangent to the $\exp_p(\partial D_r)$.
The integral curve at $p \in \exp_p(\partial D_r)$ is given by

$$\exp_p(\gamma_r(t))$$

This follows by writing

$$\exp_p\left(\lim_{s\to r} \gamma_s(t)\right)$$

and from smoothness of $\exp_p : T_pM \to M$. Consider the map $\exp_p|_{\partial D_r} : \partial D_r \to M$. The above discussion showed that the map $\gamma := \exp_p|_{\partial D_r} \circ \gamma_r : S^1 \to M$ is the integral curve of X of some $q \in \partial B_r(p)$. we have two cases

(i) If $X(q) = 0$, it follows that γ map S^1 to a point. In particular since $\gamma_r : S^1 \to \partial D_r$ is an embedding, we conclude that $\exp_p|_{\partial D_r} : \partial D_r \to M$ maps all the points to one point.

(ii) If $X(q) \neq 0$ we show that γ is an embedding. Put $\gamma(0) =: q$. Since S^1 is compact it is suffices to show that γ is an injective immersion.

For the immersion, we have $\dot\gamma = X \circ \gamma \neq 0$ since X has constant non zero length along the integral curve.

Next for the injectivity, we crucially have to use the assumption that M is orientable. To see how it might fail, look for example at the lower hemisphere and identify points on the equator with their antipodal points.
First, as γ is an integral curve and an immersion, it is locally injective and at crossings $\gamma(t_1) = \gamma(t_2)$, the tangent vectors agree as

$$\dot\gamma(t_1) = X(\gamma(t_1)) = X(\gamma(t_2)) = \dot\gamma(t_2).$$

Furthermore, as $\gamma(0) = \gamma(2\pi)$, there is a root of unity $t_0 := \frac{2\pi}{n}$ such that γ is t_0 periodic. For γ to be injective, we need to show that $n = 1$.
Now as $X|_{\partial B_r} \neq 0$, there is $\epsilon > 0$ such that on $U = B_{r+\epsilon} \backslash \overline{B}_{r-\epsilon}$, X is non zero. As M is 2 dimensional and **orientable**, there is a unique unit vector field Y orthogonal to X such that (Y, X) forms a positive basis. Up to changing to the opposite orientation, in our geodesic coordinates $Y(z)$ is a radially outgoing vector field and so the integral curves in $U \cap B_r(p)$ is given by $\nu : (r-\epsilon, r) \to M : t \mapsto \exp_p(tv)$ for $v \in T_pM$ a unit vector. By definition, the geodesics $\gamma_1(t) = \exp_p(te_1)$ and $\gamma_2(t) = \exp_p(te^{t_0 i}e_1)$ intersect at q. We have

$$\dot\gamma_1(r) = Y(q) = \dot\gamma_2(r)$$

and so by uniqueness of geodesics we must have $n = 1$.

Now we still need to see why there is $s > r$ such that $\exp_p|_{D_s}$ is still a diffeomorphism. As $\{X(q), Y(q)\} \in d\exp_p(q)T_pM$ for $q \in \partial B_r(p)$ (To see this, calculate the differential using curves.), and so $d\exp_p$ is an isomorphism on $\overline{D}_r$. By compactness, there is $\epsilon > 0$ such that $d\exp_p$ is an isomorphism on $D_{r+\epsilon}$. So we are only left with proving injectivity.

Now assume there is no $0 < \delta < \epsilon$ such that $\exp_p|_{D_{r+\delta}}$ is injective. This means we find $\delta_k \to 0$, $v_1^k \neq v_2^k \in T_pM$ unit vectors and $0 < s_1^k, s_2^k < \delta_k$ such that

$$q_k := \exp_p((r + s_1^k)v_1^k) = \exp_p((r + s_1^k)v_2^k.$$

As the unit circle is compact, there is a subsequence such that $v_1^k \to v_1$ and $v_2^k \to v_2$. We have by continuity,

$$q := \exp_p(rv_1) = \exp_p(rv_2)$$

and by injectivity established above, we have $v_1 = v_2$. But this leads to a contradiction as

$$(r + s_1^k)v_1^k \to v_1 = v_2 \leftarrow (r + s_2^k)v_2^k$$

and by Inverse function theorem, $\exp_p$ is a diffeomorphism in a neighborhood of q, so in particular injective on this neighborhood.

(c) (i) Assume that X has its only zero in $p \in M$. Then look at the set

$$S = \{r \in \mathbb{R}_+ : \exp_p : D_r \to M \text{ is a diffeomorphism } \}.$$

This set is non-empty, and by b) both open and closed, so $S = \mathbb{R}_+$ and $\exp_p : T_pM \to M$ is a diffeomorphism from $T_pM \cong \mathbb{R}^2$ and M.

(ii) It remains to treat the case where there is a second point $q \in M$ with $X(q) = 0$, i.e when $\exp_p|_{\partial D_r} : \partial D_r \to M$ map all the points into one point. Then if we identify ∂D_r with a point by an equivalence relation we get that $\exp_p$ induces an embedding from S^2 into M. Since S^2 is complete from the previous exercise sheet we know that it cannot be embedded isometrically in any 2-manifold except S^2.

4. (a) Consider a surface of revolution M (see exercise sheet 6, exercise 2) turned along the x axis. Let γ be a latitude circle. Then let C be the tangent cone on M at γ (see my picture;). Recall that two connections on M and C are obtained from the orthogonal projection of the Levi-Civita connection D of $\mathbb{R}^3$ (see exercise sheet 4 exercise 3), but in this case the two ortogonal projections

$$\pi_1 : T_p\mathbb{R}^3 \to T_pM, \quad \pi_2 : T_p\mathbb{R}^3 \to T_pC,$$

agrees if $p \in \gamma$. This show that the holonomy around γ on M is the same as the holonomy around γ on C. In particular the holonomy on the cone can be computed easily: cut the cone from the center along a line and rolled flat (see picture 1). It turns out that the holonomy is equal to the angle obtained.

(b) It follows from above. See the picture 2.