Differential Geometry II

D-MATH, FS 2015

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Supplementary Exercises

1. (a) Let $\mathbb{B}^2$ be the Poincare disk model. Show that the diameter $\gamma(t) := t$ minimize the distance between any two points on x, y and is there fore a *geodesic.*

(b) Parametrize γ by arclenght.

2. Let E be a vector bundle over a manifold M.

(a) Let $\gamma : [0,1] \to M$ be a colosed curve. Define what it means for γ to be orientation persevering, (resp. orientation-reversing) for E.

(b) Show that the property of being orientation-persevering (resp. orientation-reversing for E) is invariant under the homotopy class of γ.

(c) Define

$$w_1^E \,:\, \pi_1\left(M, p\right) \to \mathbb{Z}_2$$

via

$$w_1^E([\gamma]) := \begin{cases} 0 & \text{if } \gamma \text{ is orientation-persevering,} \\ 1 & \text{otherwise .} \end{cases}$$

Observe that w_1^E is a group homomorphism.

(d) Show that E is orientable if and only if $w_1^E \equiv 0$. w_1^E is called the first Stiefel class of E.

3. Define the oriented 2-plane bundle E_k over S^2 by gluing $B_1 \times \mathbb{R}^2$ to $B_1 \times \mathbb{R}^2$ via the map

$$\begin{array}{rccc} \phi_k : & \partial B_1 \times \mathbb{R}^2 & \to & \partial B_1 \times \mathbb{R}^2 \\ & (e^{i\theta}, (x,y)) & \mapsto & (e^{-i\theta}, R_{-k\theta}(x,y)) \end{array}$$

where $R_{-k\theta}$ is rotation by $-k\theta$, and endowing the result with the obvious orientation.

(a) Show that any $\mathbb{R}^2$ bundle over S^2 is isomorphic to E_k for some k.

(b) Prove that E_k and E_l are isomorphic as oriented 2-plane bundles only if $k = l$.

Hint: you may use the theorem that any bundle over a contractible space is trivial.

4. Let $\left(S^2, g\right)$ be the standard sphere

(a) Compute g in polar coordinate.

(b) Compute the Christoffel symbols in polar coordinates.

(c) Prove that the lines of longitude and the equator are geodesic via *b*).

5. This exercise is useful for Exercise 1 of Exercise Sheet 7. Let G be a Lie group, $X \in T_eG$. Denote with X^L the left-invariant extension of X and with X^R the right invariant extension of X. Prove

(a) $\phi_t^{X^L}(e) = \phi_t^{X^R}(e)$ for $t \in \mathbb{R}$, where ϕ^W is the flow of W.

(b) Let $\exp(tX) := \phi_1^{X^L}(e) = \phi_1^{X^R}(e)$. Then $\exp((-)X) : \mathbb{R} \to G$ is a group homomorphism called 1-parameter subgroup.

(c) $\phi_t^{X^R} = L_{\exp(tX)}$, i.e. the left multiplication by $\exp(tX)$ is the flow of X^R.

6. Let $B_\epsilon(p)$ be a geodesic ball in a Riemannian manifold, $\mathrm{d}(q) := \mathrm{dist}(p, q)$

(a) Observe that $\mathrm{d}^2 : B_\epsilon(p) \to \mathbb{R}$ is smooth for ϵ small enough.

(b) Show that for a unit speed geodesic γ in $B_\epsilon(p)$, ϵ small enough,

$$\frac{d^2}{dt^2}\mathrm{d}(\gamma(t))^2 = 2 + h_{\gamma(t)},$$

where $|h_{\gamma(t)}| \leqslant C\mathrm{d}(\gamma(t))^2$ for some C independent of γ.