D-Math | Functional Analysis 2 | FS 2015
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Serie 1

1. *[1] Let (X, d) be a compact metric space and let μ be a finite Borel measure on X. Let V be a Banach space, and let $f : X \to V$ be a continuous function. Prove that for any sequence of finite measurable partitions ξ_n satisfying

$$\max_{P \in \xi_n} \operatorname{diam}(P) \longrightarrow 0$$

the sequence of Riemann sums

$$\sum_{P \in \xi_n} f(x_P)\mu(P)$$

form a Cauchy sequence in V (for any choice of $x_P \in P$), thus establishing the existence of a Riemann integral

$$\text{R-}\int_X f d\mu \in V.$$

Hint: Use uniform continuity of f and look for two partitions late in the sequence at the common refinement $\xi_m \vee \xi_n = \{P \cap Q : P \in \xi_m, Q \in \xi_n\}$

2. Let G be a σ-compact locally compact group with left Haar measure m_G.

a) Recall that $L^1_{m_G}(G)$ is a Banach algebra: For $f_1, f_2 \in L^1_{m_G}(G)$ define their convolution in G by

$$f_1 * f_2(g) = \int f_1(h) f_2(h^{-1}g) dm_G(h).$$

Show that $f_1 * f_2(g)$ exists for m_G-almost every $g \in G$, and defines an element

$$f_1 * f_2 \in L^1_{m_G}(G)$$

with $\|f_1 * f_2\|_1 \leq \|f_1\|_1 \|f_2\|_1$.

b) Let π be a unitary representation of G on a Hilbert space H. Show that

$$f_1 \underset{\pi}{*} (f_2 \underset{\pi}{*} v) = (f_1 * f_2) \underset{\pi}{*} v$$

where $\underset{\pi}{*}$ is defined as in Definition 3.38. In other words, H is a module for the algebra $L^1_{m_G}(G)$.

[1] This exercise is exam relevant. As announced in the lecture, no solutions will be posted.