Serie 3

1. * Let $U \subset \mathbb{R}^2$ be open, $g \in H^1(U)$ such that $\Delta g = u \in L^2(U)$ and $K \subset U$ compact.

a) Show that $\|g|_K\|_{C(K)} \ll_K \|g\|_{H^2(U)}$ for any compact $K \subset U$.

b) Show that for any open $V \subset U$ with compact closure contained in U, it holds that

$$\|g\|_{H^2(V)} \ll_V \|u\|_{L^2(U)} + \|g\|_{H^1(U)}.$$

Hints:

- a) should be easy, see the Sobolev embedding theorem 3.71.
- For b), first show that for $\Delta g = u \in L^2(\mathbb{T}^2)$ and $g \in H^1(\mathbb{T}^2)$ that $\|g\|_{H^2(\mathbb{T}^2)} \ll \|u\|_{L^2(\mathbb{T}^2)} + \|g\|_{L^2(\mathbb{T}^2)}$. Then use the transfer principle Lemma 3.73 and the product rule Lemma 3.86 similiar to the proof of Theorem 3.81. [1]

2. Finish exercise 4 of Serie 2: Deduce that $\widetilde{H}^1(S) = H^1(S)$ (by choosing λ such that $S \subset (\lambda^{-1}S)_\varepsilon$).

3. Let $\mathcal{H}$ be a Hilbert space and define the space of Hilbert-Schmidt operators $\mathrm{S}_2 \subset \mathcal{B}(\mathcal{H})$ by $A \in \mathrm{S}_2$ if for an orthonormal basis $\{e_j\}$ of $\mathcal{H}$ it holds that

$$\|A\|_{\mathrm{HS}}^2 = \sum_{j,k} |\langle Ae_j, e_k\rangle|^2 < \infty.$$

a) Show that the definition of $\|\cdot\|_{\mathrm{HS}}$ is independent of the choice of basis. In fact, show that for any other orthonormal bases $\{f_j\}$ and $\{g_k\}$ it holds that

$$\|A\|_{\mathrm{HS}}^2 = \sum_{j,k} |\langle Af_j, g_k\rangle|^2$$

and that $A \in \mathrm{S}_2$ if and only if $A^* \in \mathrm{S}_2$ and $\|A^*\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}$ where A^* denotes the adjoint of A.

b) Show that S_2 forms a two-sided ideal in $\mathcal{B}$, that is, for any $A \in \mathrm{S}_2$ and any $B \in \mathcal{B}$ also $AB \in \mathrm{S}_2$ and $BA \in \mathrm{S}_2$.

c) Find a scalar-product on S_2 which induces $\|\cdot\|_{\mathrm{HS}}$. Show that S_2 is a Hilbert-space.

d) Show that S_2 is also a Banach algebra, that is, for A and B in S_2 it holds

$$\|AB\|_{\mathrm{HS}} \le \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}}.$$

e) Show that S_2 is closed in $\mathcal{B}$ if and only if $\mathcal{H}$ is finite dimensional.

[1] Numeration as in the recently updated script.

Please turn over!

f) Assume that $\mathcal{H}' = L^2(0,1)$, and let $k \in L^2(0,1) \times L^2(0,1)$. We defined the associated Hilbert-Schmidt integral operator $K : \mathcal{H}' \to \mathcal{H}'$ by

$$Kf(y) = \int f(x)k(x,y)dy.$$

Show that the space of Hilbert-Schmidt integral operators corresponds exactly to S_2 of $L^2(0,1)$. In particular, for $A \in \mathrm{S}_2$ the corresponding kernel k_A is

$$k_A(x,y) = \sum_{i,j} \langle Ae_i, e_j \rangle e_i(x) \overline{e_j(y)}.$$

g) Deduce that a Hilbert-Schmidt operator is a compact operator (for any Hilbert space $\mathcal{H}$, separable or not).