D-Math | Functional Analysis 2 | FS 2015
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Serie 4

1. * Show that not every function in $H_0^1((0,1)^d) \cap H^2((0,1)^d)$ is also in $H_0^2((0,1)^d)$.

2. Show the following connection between Hilbert Schmidt operators $\mathcal{S}_2$ and trace class operators TC.

a) If $A, B \in \mathcal{S}_2$ then $AB \in \mathrm{TC}$.

b) If $C \in \mathrm{TC}$ then there exists $A, B \in \mathcal{S}_2$ such that $C = AB$.

3. If $N(T) = \#\{n : |\lambda_n| < T\}$ denotes the count of Laplace eigenvalues on a bounded domain $\Omega \subset \mathbb{R}^d$ then $\lim_{T\to\infty} \frac{N(T)}{T^{d/2}} = c_d \operatorname{vol}(\Omega)$. Reformulate Weyl's law to say that $\lim_{n\to\infty} \frac{\lambda_n}{n^{2/d}}$ exists.

4. Show that for $f \in C_c^\infty(U)$ where $U \subset \mathbb{R}^2$ is bounded and has smooth boundary then the expansion of f as sum into Laplace eigenfunctions in $H_0^1(U)$ (see Theorem 4.31) converges in fact pointwise and uniformly so on compact subsets.

Hint: Combine a) and b) Exercise 1 of Serie 3 to bound the $\|\cdot\|_{C(K)}$ norm of an eigenfunction in terms of the L^2-norm and the eigenvalue. Then use Exercise 3 from this Serie to deduce summability properties of the coefficients of f with respect to the eigenbasis, using similiar ideas of Sobolev embedding on the torus that appeared in Chapter 3.4.1.