Serie 5

1. Let H be a normal subgroup of a discrete group G. Show that G is amenable if and only if H and G/H are amenable.

Hint: For example the $\Leftarrow$ direction can be shown by checking that the naive composition of Banach limits on $\ell^\infty(H)$ and $\ell^\infty(G/H)$ of Banach limits is well defined, and also a Banach limit on G.

2. * Show that every discrete abelian group G is amenable.

Hint: We already know that $\mathbb{Z}^d$ is amenable for all $d \geq 1$. Hence all finitely generated (fg) abelian groups are amenable by exercise 1. Now consider for a subgroup H of G the set

$$S_H = \{\Lambda \in l^\infty(G)^* : \Lambda \geq 0, \Lambda(\mathbb{1}_G) = 1 \text{ and } \Lambda \text{ is left-invariant under } H\}.$$

Use Tychonoff-Alaoglu to show that

$$\bigcap_{\substack{H<G \\ H \text{ fg}}} S_H \neq \emptyset.$$

3. Let $\{A_\tau\}_{\tau \in X}$ be a sequence of trace class operators on a separable Hilbert $\mathcal{H}$ space indexed by a probability space (X, μ) such that

$$\tau \mapsto \langle A_\tau v, w\rangle \text{ and } \tau \mapsto \|A_\tau v\|$$

are measurable for all $v, w \in \mathcal{H}$ and $\|A_\tau\|_{TR} \leq C$ is uniformly bounded. Show that $\int A_\tau d\mu$ is a trace class operator and find a formula for $\operatorname{tr}\left(\int A_\tau d\mu\right)$.

4. Assume that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two inner forms on V and V is complete with respect to $\langle\cdot,\cdot\rangle_1$. In fact, we may only assume that $\langle\cdot,\cdot\rangle_2$ is a semi-inner form and we write $S_i(v) = \sqrt{\langle v, v\rangle_i}$.

a) Show that if

$$S_2(v) \leq S_1(v)$$

for all $v \in V$ then there exists a bounded self-adjoint operator A such that $\langle v, w\rangle_2 = \langle Av, w\rangle_1$.

We may define the relative trace $\operatorname{tr}(S_2|S_1) = \operatorname{tr}(A)$, which is finite if A is trace class.

b) Let $V = H^1(\mathbb{T})$ with $S_1 = \|\cdot\|_{H^1}$ and define $S_\tau(f) = |f(\tau)|$ for $\tau \in \mathbb{T}$ which clearly comes from a semi-inner form. Show that $\operatorname{tr}(S_\tau|S_1)$ is uniformly bounded for all $\tau \in \mathbb{T}$ (e.g. by considering an orthonormal basis of $\ker(f \mapsto f(\tau))$ and complementing it to one on V).

c) Apply exercise 3 to deduce that $S_2 = \|\cdot\|_{L^2}$ has finite relative trace with respect to S_1. How did we previously write the operator A associated to $S_2 \leq S_1$?