Solution 1

1. a)

$$m(f_1)m(f_2) = \int_G \int_G f_1(h)f_2(g)dm(h)dm(g) = \int_G \int_G f_1(h)f_2(h^{-1}g)dm(h)dm(g)$$

by left-invariance of m. Since f_1 and f_2 are assumed to be absolutely integrable, the left-hand side is finite and so is the product mesure $m \times m$ over $f_1(h)f_2(h^{-1}g)$. We can therefore apply Fubini to deduce that $g \mapsto f_1 * f_2(g) = \int_G f_1(h)f_2(h^{-1}g)dm(h)$ exists m-a.e.. Taking absolute values of $f_1 * f_2$, and taking them into the inner integral, we see that $\|f_1 * f_2\|_1 \leq m(|f_1|)m(|f_2|) = \|f_1\|_1\|f_2\|$.

b) The integral $f *_\pi v$ are defined using Frechet-Riesz (see Lemma 3.37). To make everything formel, one should take scalar product with an arbitrary vector $v \in H$ (in particular, we really use the scalar-valued Fubini in the last step).

We write out the left-hand side first.

$$f_1 *_\pi (f_2 *_\pi v) = \int_G f_1(h)\pi(h) \left(\int_G f_2(g)\pi(g)vdm(g) \right) dm(h) = \int_G \int_G f_1(h)f_2(g)\pi(h)\pi(g)vdm(g)dm(h)$$

Since π is a representation and using left-invariance of the integral over g,

$$= \int_G \int_G f_1(h)f_2(g)\pi(hg)vdm(g)dm(h) = \int_G \int_G f_1(h)f_2(h^{-1}g)\pi(g)vdm(g)dm(h)$$

Changing the order of integration (a second application of Fubini),

$$= \int_G \left(\int_G f_1(h)f_2(h^{-1}g)dm(h) \right) \pi(g)vdm(g) = (f_1 * f_2) *_\pi v.$$