Solution 2

1. We give both arguments. Let us remark that previously, the weak-derivative was only defined for open subsets (Definition 3.50) and using functions with compact support which would imply the vanishing of boundary terms. For $\mathbb{T}^d$ periodicity will imply that partial integration works without additional terms, that is the formula of Definition 3.50 holds for all smooth f and φ on $\mathbb{T}^d$. In particular, also Lemma 3.52 holds, that is, if $f \in H^k(\mathbb{T}^d)$ then f_α agrees with the weak derivative $\boldsymbol{\partial}_\alpha f$. We therefore only have to show that if $\boldsymbol{\partial}_\alpha f$ exists for all $\|\alpha\| \leq k$ then $f \in H^k(\mathbb{T}^d)$.

a) We rewrite

$$\langle f, \boldsymbol{\partial}_\alpha \varphi \rangle = (-1)^{\|\alpha\|} \langle \boldsymbol{\partial}_\alpha f, \varphi \rangle$$

in terms of the Fourier Series (notation as in Proposition 3.46)

$$\sum f_n (2\pi n)^\alpha \varphi_n = (-1)^{\|\alpha\|} \sum (\boldsymbol{\partial}_\alpha)_n f \varphi_n$$

from which we deduce that $f_n(2\pi n)^\alpha \in \ell^2(\mathbb{Z}^d)$ (uniqueness of fourier coefficients). Taking the sum over all $\|\alpha\| \leq k$ we see from this that $f_n \|n\|_2^k \in \ell^2(\mathbb{Z}^d)$. By Proposition 3.47 this is equivalent to $f \in H^k(\mathbb{T}^d)$.

b) For the mollifier argument, we only need to note that

$$J_\varepsilon * f(x) = \int J_\varepsilon(x-y) f(y) dy = \langle J_\varepsilon(x - \cdot), f \rangle.$$

Since $J_\varepsilon * f$ is smooth with α-derivative equal to $\partial_\alpha(J_\varepsilon * f) = (\partial_\alpha J_\varepsilon) * f$ (by use of dominated convergence) the weak-derivative formula reads

$$(\partial_\alpha J_\varepsilon) * f = (-1)^{\|\alpha\|} J_\varepsilon * \boldsymbol{\partial}_\alpha f.$$

Since $J_\varepsilon * g \to g$ in L^2 for $g \in L^2$ as $\varepsilon \to 0$ this implies that $((J_\varepsilon * f)_\alpha)_{\|\alpha\| \leq k} \to (\boldsymbol{\partial}_\alpha f)_{\|\alpha\| \leq k}$ in $(L^2)^{K(k)}$ and by definition of the Sobolev space, $f \in H^k(\mathbb{T}^d)$.

2. Note the reason, why we cannot argue just as in the previous exercise using mollifiers. $J_\varepsilon(x - \cdot)$ has support in $\overline{B_\varepsilon(x)}$, which for x close to the boundary of S does not lie inside S. In particular, $J_\varepsilon(x - \cdot) \notin C_c^\infty(S)$ and we cannot use J_ε as a test function for the weak derivative. To overcome this problem we thicken S, by dilating with a factor λ. Exercise a) will show that this new function is still close to f. Exercise b) will be the analogue of the previous calculation for the torus. In the next serie we combine both steps to again conclude that the existence of all weak derivatives up to order k imply that $f \in H^k(S)$.

a) Apply exercise 2 of the same series to the diffeomorphism $x \mapsto \lambda x$ to see that f^λ has finite Sobolev norm on $\lambda^{-1} S$. By exercise 1, we also have a Sobolev function $f^\lambda|_S$ on S. Of course, both exercise 1 and exercise 2 we only calculated for H^1 (and not $\tilde{H}^1$), and one should verify those properties also for functions for which a priori only the weak derivatives exist. We omit this, as in fact the convergence claimed will only be used for H^1 functions, and the existence of a bounded restriction operator is trivial.

If $g \in C_c(S)$ then $g^\lambda \to g$ in C_c since for any uniform continuity pair (ε, δ) of g, we find λ sufficiently close to 1 such that $\lambda x \in B_\delta(x)$. As S is bounded, we also have $g^\lambda \to g$ in $L^2(S)$. For arbitrary $f \in L^2(S)$ take a sequence $g_n \to f$ of continuous functions (which exists by density of C_c in L^2) and apply the 3ε argument. We conclude that $f^\lambda \to f$ in $L^2(S)$. The α-weak derivative of f^λ is $\lambda \boldsymbol{\partial}_\alpha f^\lambda$ for $\|\alpha\| = 1$. By the previous argument we have $(\boldsymbol{\partial}_\alpha f)^\lambda \to \boldsymbol{\partial}_\alpha f$ and thus also $\boldsymbol{\partial}_\alpha f^\lambda = \lambda(\boldsymbol{\partial}_\alpha f)^\lambda \to \boldsymbol{\partial}_\alpha f^\lambda$ (which we may again deduce by first checking this for continuous functions) which concludes H^1 convergence.

b) Argument as in the last exercise, which is now possible as $J_\varepsilon(x - \cdot)$ is in $C_c^\infty(S_\varepsilon)$