Solution 3

1. The inclusion $H^1 \subset \tilde{H}^1$ is clear. We also note that the convergence $f^\lambda \to f$ shown in Serie 2 is only needed for H^1 function: Indeed, for ε_0 there exists $\lambda < 1$ such that $(\lambda^{-1}S)_\varepsilon \supset S$ for all $\varepsilon < \varepsilon_0$. This implies that for $f \in \tilde{H}^1(S)$ we have $f^\lambda|_S \in \tilde{H}^1(S)$. Thus $J_\varepsilon * f^\lambda|_S \to f^\lambda|_S$ in the $\|\|_{H^1(S)}$-norm since the weak derivatives of $f^\lambda|_S$ exist on all of S by assumption on λ and part b) of the exercise in Serie 2. As $J_\varepsilon * f^\lambda|_S$ is a smooth function, $f^\lambda|_S$ lies in $H^1(S)$. Letting $\lambda \to 1$, we can apply part a) of the exercise in Serie 2 to deduce that $f \in H^1(S)$.

2. Note that by definition, the support of A can only contain countable many basis vectors. In particular, we may restrict to the separable Hilbert space $\mathcal{H}$ spanned by the support of A.

a) Any element $v \in \mathcal{H}$ can be decomposed as $v = \sum \langle v, e_i \rangle e_i$ and $\|v\|^2 = \sum |\langle v, e_i \rangle|^2$ because the $\{e_i\}$ are orthonormal (see the Hilbert space chapter in the script). Applying this to $\|A\|_{\mathrm{HS}}^2$ we see that

$$\|A\|_{\mathrm{HS}}^2 = \sum_{i,j} |\langle Ae_j, e_i \rangle|^2 = \sum_j \left(\sum_i |\langle Ae_j, e_i \rangle|^2 \right) = \sum_j \|Ae_j\|^2.$$

This was independent of the basis choosen in the second argument, thus $\sum_{i,j} |\langle Ae_j, f_i \rangle|^2 = \sum_{i,j} |\langle Ae_j, e_i \rangle|^2$ for any other orthonormal basis $\{f_i\}$. On the other hand,

$$\sum_{i,j} |\langle Ae_j, e_i \rangle|^2 = \sum_{i,j} |\langle e_j, A^* e_i \rangle|^2 = \sum_i \|A^* e_i\|^2$$

which is now independent of the basis in the first argument, and so also $\sum_{i,j} |\langle Ag_j, e_i \rangle|^2 = \sum_{i,j} |\langle Ae_j, e_i \rangle|^2$ for any other orthonormal basis $\{g_j\}$, which altogether implies the claim, and also shows that $\|A\|_{\mathrm{HS}} = \|A^*\|_{\mathrm{HS}}$.

b) From the description

$$\|A\|_{\mathrm{HS}}^2 = \sum_j \|Ae_j\|^2$$

we see that

$$\|BA\|_{\mathrm{HS}}^2 = \sum_j \|BAe_j\|^2 \le \|B\|_{\mathrm{OP}}^2 \|A\|_{\mathrm{HS}}^2.$$

Since the adjoint of AC is $C^* A^*$ we also infer that

$$\|AC\|_{\mathrm{HS}} = \|(AC)^*\|_{\mathrm{HS}} \le \|C^*\|_{\mathrm{OP}}^2 \|A\|_{\mathrm{HS}}^2$$

c) Fix an orthonormal basis $\{e_i\}$ on $\mathcal{H}$. Define

$$\langle A, B \rangle_{\mathrm{HS}} = \sum_{i,j} \langle Ae_i, e_j \rangle \langle e_j, Be_i \rangle$$

which is bilinear by bilinearity of $\langle \cdot, \cdot \rangle$ and linearity of the elements of S_2. Clearly, this form induces $\| \cdot \|_{\mathrm{HS}}$ Note that there is a linear map into $l^2(\mathbb{N} \times \mathbb{N})$ by mapping A to the sequence $(\langle Ae_i, e_j \rangle)$, and is isometric between the newly defined inner form and the l^2 scalar product. This map is also surjective since for any l^2-sequence a_{ij} we may associate the operator $P : v \mapsto \sum_{i,j} a_{i,j}\langle v, e_i\rangle e_j$ and the corresponding sequence is

$$\langle Pe_k, e_l\rangle = \sum_{i,j} a_{i,j}\langle\langle e_k, e_i\rangle e_j, e_l\rangle = \sum_{i,j} a_{i,j}\langle e_k, e_i\rangle\langle e_j, e_l\rangle = \sum_{i,j} a_{i,j}\delta_{ki}\delta_{jl} = a_{k,l}$$

which implies that P really defines a Hilbert-Schmidt operator. From this we infer that S_2 is also complete with respect to $\|\cdot\|_{\mathrm{HS}}$.

d)

$$\|AB\|_{\mathrm{HS}}^2 = \sum_{i,j} |\langle ABe_j, e_i\rangle|^2 = \sum_{i,j} |\langle Be_j, A^*e_i\rangle|^2 \leq \sum_{i,j} \|Be_j\|^2\|A^*e_i\|^2$$
$$= \sum_j \|Be_j\|^2 \sum_i \|A^*e_i\|^2 = \|B\|_{\mathrm{HS}}^2\|A^*\|_{\mathrm{HS}}^2 = \|A\|_{\mathrm{HS}}^2\|B\|_{\mathrm{HS}}^2$$

e) Closedness is equivalent to say that any null-sequence is square integrable, which in particular implies the claim. Let us check this: Note that any norm on a finite-dimensional space is complete which shows one direction (if $\mathcal{H}$ is finite dimensional, then so is the space of operators $\mathcal{B}(\mathcal{H})$). Assume that $\mathcal{H}$ is infinite dimensional then completeness would imply that any bounded sequence is square integrable as follows: Let b_i be a sequence then we may define the finitely supported (and thus square integrable) sequence $b_i|_N$ by $b_i|_N = b_i$ for $i < N$ and zero otherwise. Embed these sequences on the space of sequences of $\mathbb{N} \times \mathbb{N}$ diagonally (e.g. $a_{ij} = a_i\delta_{ij}$ for a sequence a_i). Then the associated operator to a sequence a_i is $v \mapsto \sum a_i\langle v, e_i\rangle e_i$ and its operator norm is easily checked to be $\sup_i |a_i|$. In particular, any bounded sequence defines a bounded operator. It is also clear that $a_i|_N \to a_i$ in ℓ^∞ if a_i is a null-sequence, and thus the corresponding operators would converge in norm topology. But the operator associated to a_i does not define a Hilbert-Schmidt operator unless a_i itself is square-integrable.

f) Let e_i be an onb of $L^2(0,1)$ and let $k \in L^2 \times L^2$. Using the decomposition on both factors, $k = \sum a_i b_j e_i e_j$, and both a_i and b_j are square integrable and thus $a_i b_j$ is square-integrable on $\mathbb{N} \times \mathbb{N}$. Note that the associated Hilbert-Schmidt operator P is exactly K since

$$Kf(x) = \langle f, k(\cdot, x)\rangle = \langle f, \sum_{ij} a_i b_j e_i e_j(x)\rangle = \sum a_i b_j \langle f, e_i\rangle e_j(x) = Pf(x).$$

g) Either use your knowledge that a Hilbert-Schmidt integral operator is compact, and identifying $\mathcal{H}$ with $L^2(0,1)$ by mapping one orthonormal basis to another. Or use that an operator is compact if it can be approximated by operators with finite dimensional range in norm topology, which we clearly can by the argument of part e).