Solution 4

1. a) Let $A, B \in \mathcal{S}_2$. Then also $A^* \in \mathcal{S}_2$ so that

$$|\sum_n \langle ABe_n, f_n\rangle| = |\sum \langle Be_n, A^* f_n\rangle| = |\sum_{n,m} \langle Be_n, \langle A^* f_n, e_m\rangle e_m\rangle|$$

$$= |\sum_{n,m} \langle Be_n, e_m\rangle \langle A^* f_n, e_m\rangle| \leq \sqrt{\sum_{n,m} |\langle Be_n, e_m\rangle|^2} \sqrt{\sum_{n,m} |\langle A^* f_n, e_m\rangle|^2} \leq \|B\|_{HS}\|A\|_{HS}$$

for any orthonormal basis e_n and f_n, so that also $\|AB\|_{TC} = \sup_{e_n,f_n} |\sum_n \langle ABe_n, f_n\rangle| \leq \|B\|_{HS}\|A\|_{HS}$

b) We have seen in the lecture that we can decompose a compact operator C into an isometry U and a compact positive self-adjoint operator A so that $C = UA$. Not that if C is trace class, it is also compact and we can apply the spectral theorem of compact self-adjoint operators to A. This also permits us to take the squareroot $\sqrt{A}$ of A, the operator defined by $e_n \mapsto \sqrt{\lambda_n} e_n$ where e_n is an eigenbasis of A. Then $C = (U\sqrt{A})\sqrt{A}$ and it remains to check that the squareroot of a positive diagonalizable trace class operator is Hilbertschmidt. But then again, we have seen in the lecture that in that case the trace norm equals the sum of the eigenvalues (actually, we only seen that the trace functional is equal to that, but by positivity it equals the trace norm). Using that the trace is norm independent (for which we worked hard in the lecture)

$$\|\sqrt{A}\|_{HS}^2 = \sum \langle \sqrt{A} e_k, e_k\rangle^2 = \sum \lambda_k = \operatorname{tr}(A)$$

2. Weyls law gives that for any $\varkappa > 0$ and for sufficiently large T (depending on $\varkappa$) we have

$$c_d T^{d/2}(1-\varkappa) \leq N(T) \leq (1+\varkappa) c_d T^{d/2}.$$

We now set $T_n = (n/c_d)^{2/d}$ then let $\varepsilon > 0$ and apply $T = T_n(1 \pm \varepsilon)$ to above inequalities so that

$$n(1-\varkappa)(1+\varepsilon)^{d/2} \leq N(T_n(1+\varepsilon))$$

and

$$n(1+\varkappa)(1-\varepsilon)^{d/2} \geq N(T_n(1-\varepsilon)).$$

We can choose ε such that $(1-\varkappa)(1+\varepsilon)^{d/2} > 1$ and $(1+\varkappa)(1-\varepsilon)^{d/2} < 1$ such that $\varepsilon \to 0$ as $\varkappa \to 0$. With this choice

$$N(T_n(1-\varepsilon)) < n < N(T_n(1+\varepsilon))$$

which implies that

$$T_n(1-\varepsilon) < \lambda_n < T_n(1+\varepsilon)$$

where λ_n denotes the nth eigenvalue. Divide by T_n so that and take the limit over n,

$$\lim_n \frac{\lambda_n}{T_n} \in [1-\varepsilon, 1+\varepsilon]$$

and finally let $\varkappa \to 0$.

3. Let K be an arbitary compact subset of U for which we want to show uniform convergence. Combining a) and b) of exercise 1 serie 3 for $g \in C^\infty(U)$ and $K \subset V \subset U$ where V is open and precompact in U,

$$\|g|_K\|_\infty \ll \|\Delta g\|_{L^2(V)} + \|g\|_{H^1(V)}.$$

If now $f_n \in H_0^1(U)$ is an Laplace eigenfunction then we may bound $\|\Delta f_n\|_{L^2(V)} + \|f_n\|_{H^1(V)}$ by $\|\Delta f_n\|_{L^2(U)} + \|f_n\|_{H^1(U)}$. For $\varphi \in C_c^\infty$ we can integrate by part so that $\langle f_n, \varphi\rangle_1 = -\langle \Delta f_n, \varphi\rangle_2 = |\lambda_n|\langle f_n, \varphi\rangle_2$ (where the norm H^1 is taken to be that induced by $\langle\cdot,\cdot\rangle_1$ in notation from the script) which is a closed equation and thus also holds in all of H_0^1. Therefore $\|f_n\|_{H^1(U)} = \sqrt{|\lambda_n|}\|f_n\|_{L^2(U)}$. Clearly, $\|\Delta f_n\|_{L^2(U)} = |\lambda_n|\|f_n\|_{L^2(U)}$ and for n sufficiently large therefore

$$\|f_n|_K\|_\infty \ll |\lambda_n|\|f_n\|_{L^2(U)}.$$

Let $f \in C_c^\infty(U)$ and consider the two expansions in an orothonormal eigenbasis f_n:

$$f = \sum \langle f, f_n\rangle f_n \quad (S) \quad \text{and } \Delta f = \sum \langle \Delta f, f_n\rangle f_n = \sum \lambda_n \langle f, f_n\rangle f_n$$

where convergence is in L^2 and we have used partial integration in the last equality. Note that it is precisely here that we use that f has compact support to make integration by parts work (and we refer here to exercise 1). The last decomposition implies that

$$\sum |\langle f, f_n\rangle| \le \|\Delta f\|_{L^2} \sqrt{\sum \lambda_n^{-2}} \qquad (M)$$

by Cauchy-Schwarz applied to the two sequences $|\lambda_n|\langle f, f_n\rangle$ and λ_n^{-1}. By exercise 3 we know that (recall that $d = 2$) $\lambda_n \ge \frac{1}{2}n$ for sufficiently large n which implies convergence of $\sqrt{\sum \lambda_n^{-2}}$. We have

$$\|f|_K\|_\infty \le \sum |\langle f, f_n\rangle| \|f_n|_K\|_\infty \ll \sum |\langle f, f_n\rangle||\lambda_n| = \sum |\langle f, \Delta f_n\rangle|$$

having used that $\|f_n\|_{L^2(U)} = 1$. Applying (M) to Δf instead of f gives therefore

$$\|f|_K\|_\infty \ll \|\Delta^2 f\|_{L^2}$$

and that the sum (S) converges absolutely.