Exercise sheet 1

Exercise 1.1 Consider the following *binomial market* on some probability space $(\Omega, \mathcal{F}, P)$:

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = 1 + r, \qquad S^1 = Y,$$

where Y takes the value $1 + u$ with probability $p \in (0, 1)$ and $1 + d$ with probability $1 - p$.

Assume that $u = r > d$ and construct an arbitrage opportunity.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be given by $\Omega = \{\omega_u, \omega_m, \omega_d\}$, $\mathcal{F} = 2^\Omega$ and $P(\{\omega_i\}) = p_i \in (0, 1)$ for $i \in \{u, m, d\}$ and real numbers $u > m > d > -1$. Assume that $u > r > d$.

Characterize the set $\mathcal{P}$ of equivalent martingale measures of the market consisting of assets given by

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = 1 + r, \qquad S^1 = Y,$$

where $Y(\omega_i) = 1 + i$ for $i \in \{u, m, d\}$.

Hint: Any probability measure on $(\Omega, \mathcal{F})$ can be seen as a point in $[0, 1]^3$ and the set $\mathcal{P}$ can then be parametrized as a line segment in $[0, 1]^3$.

Remark: This is called a *trinomial model*.

Exercise 1.3 In this exercise we consider the probability space $(\Omega, \mathcal{F}, P)$ and a market given by

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = e^r, \qquad S^1 = e^Y,$$

where Y follows a standard normal distribution under P.

(i) Show that P^* given by

$$\frac{dP^*}{dP} = \exp\left(-\left(\frac{1}{2} - r\right)Y - \frac{\left(\frac{1}{2} - r\right)^2}{2}\right)$$

is an equivalent martingale measure (EMM).

(ii) Let C^{call} be the payoff of a call option on S^1 with strike K, i.e.,

$$C^{\text{call}} = (S^1 - K)^+.$$

Calculate the arbitrage-free price of C^{call} under the measure P^*, i.e., calculate

$$E^*\left[\frac{C^{\text{call}}}{S^0}\right].$$

Remark: This is a special case of the price of a call option in the Black-Scholes model. However, since the payoff is not attainable in this market, there is no unique price in this market.

Exercise 1.4 Assume we are working in some market which is free of arbitrage and has the risk-free interest rate r. Also assume that all payoffs described below are traded in this market.

(i) A (long) forward contract is an agreement at time 0 which allows an agent to buy an asset at time 1 at a predetermined *delivery price* K. In other words, a forward contract on asset i corresponds to the payoff

$$S^i - K$$

at time 1. No asset is exchanged at time 0.

Determine the delivery price of asset i.

(ii) Consider the payoffs

$$C^{\text{call}} = (S^i - K)^+ \quad \text{and} \quad C^{\text{put}} = (K - S^i)^+$$

corresponding a call and a put option with strike K respectively.

Show the *put-call parity*

$$\pi(C^{\text{call}}) = \pi(C^{\text{put}}) + \pi^i - \frac{K}{1+r}$$

by constructing a suitable portfolio.