Exercise sheet 2

Exercise 2.1 Consider the following trinomial market: let $(\Omega, \mathcal{F}, P)$ be given by $\Omega = \{\omega_u, \omega_m, \omega_d\}$, $\mathcal{F} = 2^\Omega$ and $P(\{\omega_i\}) = p_i \in (0,1)$ for $i \in \{u, m, d\}$ and real numbers $u > m > d > 0$. Also assume that $u > 1 + r > d$.

With this setup let the assets evolve as follows:

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = 1 + r, \qquad S^1 = Y,$$

where $Y(\omega_i) = i$ for $i \in \{u, m, d\}$.

Calculate the super-replication price of a call option with strike K as a function of K. That is, calculate $\Pi_{\sup}(C^{\text{call}})$ where $C^{\text{call}} = (S^1 - K)^+$ as a function of K.

Remark: This is the same market as in Exercise 1.2, but the definition of Y is changed slightly.

Exercise 2.2 We have previously considered binomial and trinomial markets. This can be generalized to any finite number of outcomes. For example, fix $n \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, P)$ be given by $\Omega = \{\omega_i : i = 1, \ldots, n\}$, $\mathcal{F} = 2^\Omega$ and $P(\{\omega_i\}) = p_i \in (0,1)$ for $i = 1, \ldots, n$. Define the market by

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = 1 + r, \qquad S^1 = Y,$$

where $Y(\omega_i) = y_i$ for some $y_i \in [0, \infty)$. As before, define X^1 as the discounted asset price

$$X^1 = \frac{S^1}{1 + r}.$$

Assume that

$$\max_{i=1,\ldots,n} y_i > 1 + r > \min_{i=1,\ldots,n} y_i.$$

(i) Verify that

$$M(X^1, P) = \left\{ E_Q\left[\frac{S^1}{1+r}\right] : Q \approx P \right\} = \operatorname{ri}\Gamma(P \circ (X^1)^{-1}),$$

where $\Gamma(\mu) = \operatorname{conv}(\operatorname{supp}\mu)$.

(ii) Show that $1 \in \operatorname{ri}\Gamma(P \circ (X^1)^{-1})$ to conclude that the market is free of arbitrage.

Exercise 2.3 Consider the model studied in Exercise 1.3, i.e., a market defined by

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = e^r, \qquad S^1 = e^Y,$$

defined on some probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F} = \sigma(Y)$ and Y is a standard normal random variable under P.

(i) Show that the EMM P^* from Exercise 1.3 is not unique by showing that the call option from the same exercise is not attainable.

(ii) Suppose now that S^1 is not necessarily distributed as above, but instead has some other distribution. Assume further that the market is free of arbitrage. Show that the market is complete if and only if S^1 is binomially distributed.

Exercise 2.4 We again consider the model from Exercise 1.3. If you want, you may assume that $\Omega = \mathbb{R}$.

Define

$$\Pi^b(C^{\text{call}}) := \left\{ E^b\left[\frac{C^{\text{call}}}{e^r}\right] : S^1 \text{ is binomially distributed under } P^b, E^b\left[\frac{S^1}{e^r}\right] = \pi^1 \right\}.$$

This is the set of arbitrage free prices under some measure for which S^1 is binomially distributed.

The goal of this exercise is to show that

$$\Pi^b(C^{\text{call}}) \subseteq \left[\Pi_{\inf}(C^{\text{call}}), \Pi_{\sup}(C^{\text{call}})\right].$$

(i) Construct a sequence of martingale measures absolutely continuous to P that converges weakly to a martingale measure under which S^1 is binomially distributed.

(ii) Construct convex combinations of this sequence and P^* found in Exercise 1.3 to show the inclusion above.

Exercise 2.5 It is known that for any market which is free of arbitrage it holds that

$$\left(\pi^1 - \frac{K}{S^0}\right)^+ \leq \Pi_{\inf}(C^{\text{call}})$$

and

$$\Pi_{\sup}(C^{\text{call}}) \leq \pi^1.$$

Show that these inequalities are equalities for the market from Exercise 1.3.