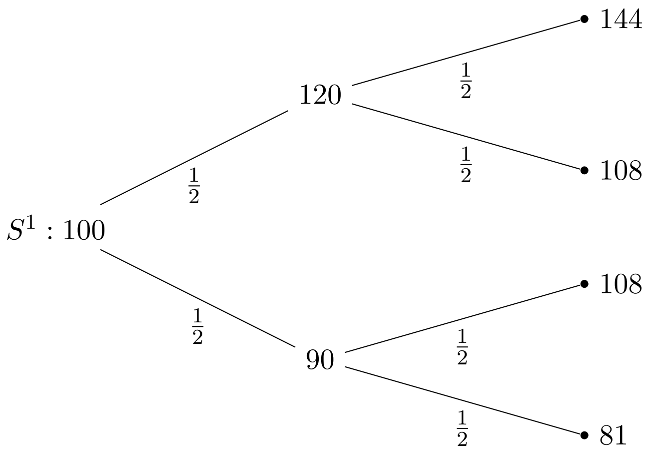
Exercise sheet 3

Exercise 3.1 In Example 1.63 we used the fact that for $F \in L^1([0,1])$, then

$$F_n := (F^+ \wedge n)I_{[\frac{1}{n},1]} - F^- \underset{n\to\infty}{\longrightarrow} F.$$

Prove that this is indeed true.

Exercise 3.2 Consider a market where the risky asset evolves as in the tree below and the risk-free interest rate is $r = 0\%$.



Find a strategy which replicates the payoff of a call option with strike 90.

Exercise 3.3 Consider a general multi-period market and denote by $C(t, K)$ the payoff $(S_t^1 - K)^+$ at time t. Call t the maturity of the option. Assume that these options are traded, that $r \geq 0$, and that the market is free of arbitrage.

Fix K and show that the price of such call options is non-decreasing as a function of maturity.

Exercise 3.4 The binomial market from above can be extended to any arbitrary number of periods T by settings

$$S_t^0 = (1 + r)^t$$

and defining the risky asset with initial price 1 by its returns

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}},$$

which are assumed to only take two values a, b with

$$-1 < a < b$$

in each period. For further details on the construction of such a model, see Chapter 5.5. Assume that the market is free of arbitrage.

Let H be a contingent claim of the form $H = h(S_T^1)$. The (undiscounted) arbitrage free price of such a claim at time k can then be written as $V_k^H = v(k, S_k^1)$. It can be shown that the function $v(k, S_k^1)$ fulfills the following backward recursion formula:

$$v(k,x) = \frac{qv(k+1, x(1+b)) + (1-q)v(k+1, x(1+a))}{1+r}, \tag{1}$$

where

$$q := \frac{r-a}{b-a}$$

and

$$v(T, x) = h(x).$$

(a) Using the above formula, derive an explicit formula for the replicating portfolio as a function of S^1, i.e., $\xi_t^1 = f(t, S_{t-1}^1)$.

(b) Implement (1) in a computer to see how the price changes with maturity and compare to Exercise 3.3. The parameters chosen do not matter as long as $-1 < a < r < b$.

One example would be $S_0^1 = 100$, $a = 0.2$, $b = -0.1$, $r = 0.05$, $K = 110$, $H = (S_T^1 - K)^+$, and T up to 100.