Exercise sheet 4

Exercise 4.1 Instead of calculating derivative prices explicitly, it can sometimes be beneficial to simulate the market instead. The idea is to simulate a large number of outcomes of the stock prices, then calculate the payoff in each scenario, and finally find the average over all outcomes. This is actually an approximation of the expected value, which we know gives the price, and is called Monte Carlo simulation.

Employ Monte Carlo simulation to find the price of the market from Exercise 3.4. Note that the simulation will have a slow convergence rate so be prepared to choose a large number of simulations.

Exercise 4.2 In this exercise we will see an example of why the finiteness of our finite period model is crucial.

We are going to construct a so called doubling strategy. The idea is to increase the size of bets whenever the asset loses value. This is done in such a way that the value is positive if the stock appreciates, and in such a scenario the process is stopped.

Consider the binomial model from Exercise 3.4 with $a = b = 0.5$ and $r = 0$. Define the trading strategy as

$$\xi_t = \frac{1}{S_{t-1}} 2^{t-1} 1_{\{t \leq \tau\}}$$

with $\tau = \inf\{t | R_t = b\}$.

(a) Run Monte Carlo simulations with increasingly many time steps to get an idea of the value of such a portfolio. What did you find?

(b) Instead of calculating the value of the strategy, calculate/simulate the biggest loss to see how it changes with the number of time periods. What is this relationship?

(c) Argue why it seems unreasonable to directly extend the results from finite periods to infinitely many periods.

Remark: A doubling strategy is sometimes called a martingale strategy.

Exercise 4.3 Consider a general arbitrage free model with a unique EMM P^*. Let $C(K) = (S_T^1 - K)^+$ be a call option payoff with strike K and denote by $V_0(C(K))$ its price at time 0. Assume S^0 is deterministic.

(a) Show that if $K_3 = \lambda K_1 + (1 - \lambda) K_2$ for some strike prices K_1, K_2, and K_3, as well as $\lambda \in [0, 1]$, then

$$V_0(C(K_3)) \leq \lambda V_0(C(K_1)) + (1 - \lambda) V_0(C(K_2)).$$

(b) Denote by $\mu = P^* \circ (X_T^1)^{-1}$ the law of X_T^1. Show that

$$\lim_{\epsilon \searrow 0} \frac{V_0(C(K)) - V_0(C(K-\epsilon))}{\epsilon} = -\frac{1}{S_T^0}\mu\left(\left[\frac{K}{S_T^0}, \infty\right)\right).$$

Exercise 4.4 An American option is an option which can be exercised at any point up until its maturity. If the option is exercised at time t, the owner gets the discounted payoff U_t. We call U the discounted payoff process. Assume the filtration is generated by U and that we have a unique pricing measure. The value of the option is then assumed to be

$$V_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E[U_\tau | \mathcal{F}_t] \tag{1}$$

where $\mathcal{T}_{t,T}$ is the set of stopping times with values in $\{t, \ldots, T\}$.

(a) Show that V is the smallest supermartingale that dominates U, i.e., with the property $V_t \geq U_t$ almost surely for $t = 0, \ldots, T$.
Hint: Show that the set $\{E[U_\tau|\mathcal{F}_t] | \tau \in \mathcal{T}_{t,T}\}$ is upward directed and use Theorem A.33. (Description of upward directedness is found in the theorem.)

Define $\bar{V}$ recursively according to

$$\begin{aligned} \bar{V}_T &= U_T, \\ \bar{V}_t &= \max\{U_t, E[\bar{V}_{t+1}|\mathcal{F}_t]\} \quad t < T. \end{aligned}$$

(b) Show that $\bar{V}$ is the smallest supermartingale that dominates U and conclude that $\bar{V} = V$ almost surely.

(c) Define the stopping times $\sigma_t = \inf\{n \geq t | U_n = \bar{V}_n\}$ (you may assume that these are indeed stopping times). Show that σ_t is a maximizer in (1) by first showing that $\bar{V}^{\sigma_t} = (\bar{V}_{s \wedge \sigma_t})_{s=t,\ldots,T}$ is a martingale.