Exercise sheet 5

Exercise 5.1 Consider a one-step trinomial model with $\mathcal{F}_0$ trivial and returns u, m, and d for which it holds that $u > m > d$. Assume that $S_t^0 = (1+r)^t$ for some fixed $r > -1$. Let H be a payoff at time 1 and denote by $(V_t)_{t=0,1}$ the corresponding superreplication price process.

(a) Find a uniform Doob decomposition of V if $u = -d$ and $m = r = 0$.

(b) Show that the decomposition is in general not unique.

Exercise 5.2 Extend the model above to multiple time periods by letting the returns for all periods be iid.

(a) Using elementary methods, prove that there exist an adapted, non-decreasing process B and a predictable process ξ such that

$$V_t = V_0 + G_t(\xi) - B_t.$$

(b) Write an algorithm for finding these processes.

Exercise 5.3 Let $X = \mathbb{R}^d_+$. A preference relation $\succeq$ on X is called monotonic if for any $x, y \in X$ such that $y^i > x^i$ for all components i, then $y^i \succ x^i$. It is called strongly monotonic if $y^i \geq x^i$ for all components i and $y \neq x$ imply $y \succ x$.

Let $\succeq$ be a continuous, strongly monotonic preference relation on X. The goal is to show that $\succeq$ has a continuous, numerical representation.

(a) Denote by e the vector with all components equal to 1. Prove that for each $x \in X$ there exists a unique $\alpha(x) \in \mathbb{R}$ such that $\alpha(x)e \sim x$.

(b) Define $U(x) = \alpha(x)$ and show that U is a continuous, numerical representation of $\succeq$.

A monotonic preference relation on X is called *homothetic* if

$$x \sim y \Longrightarrow \beta x \sim \beta y, \quad \forall \beta \geq 0.$$

(c) Show that $\succeq$ is homothetic if and only if it admits a utility function u that is homogeneous of degree one, i.e., $u(\beta x) = \beta u(x)$ for all $\beta \geq 0$.

Exercise 5.4 Imagine the possibility to obtain one of the following three monetary prices:

First prize	Second prize	Third prize
2'500'000 CHF	500'000 CHF	0 CHF

Consider the following four scenarios:

- L_1: Win the second prize with certainty.
- L_1': 10 % chance of winning first prize, 89 % of second, and 1 % of third.
- L_2: 11 % chance of winning second prize and 89 % of third.
- L_2': 10 % chance of winning first prize and 90 % of winning third.

Individuals facing the decision between L_1 and L_1' as well as between L_2 and L_2' commonly express the preferences $L_1 \succ L_1'$ and $L_2' \succ L_2$. Model these outcomes and choices mathematically, and show that this is not consistent with the existence of a von Neumann-Morgenstern representation of $\succ$.