Exercise sheet 6

Exercise 6.1 Consider a model with risk free interest rate $r = 0$ and returns

$$R_t^i = \frac{S_t^i - S_{t-1}^i}{S_{t-1}^i}$$

with R_t^i independent over time and distributed as R^i. Denote by R or R_t the vector of such random variables. Let $\mathcal{F}_t$ be the sigma algebra generated by S up until time t.

Throughout the exercise all ξ will be assumed to be predictable. Let $U : \mathbb{R} \to \mathbb{R}$ be some concave utility function which is bounded from above and define $v(t, x)$ as

$$v(t, x) = \sup_{\xi \in \mathbb{R}^{d \times T}} E[U(X_T(\xi, t, x))], \quad \forall x \in \mathbb{R}, t = 0, \ldots, T,$$

where

$$X_T(\xi, T, x) = x,$$
$$X_T(\xi, t, x) = x + \sum_{s=t+1}^{T} \xi_s \cdot R_s.$$

(a) Show that

$$v(t, x) = \sup_{\alpha \in \mathbb{R}^d} E[v(t+1, x + \alpha \cdot R)], \quad \forall x \in \mathbb{R},$$

where $t < T$.

Hint: You may assume that for any t, x, α, and $\delta > 0$ there exists a ξ^δ such that

$$v(t+1, x + \alpha \cdot R_{t+1}) \leq E[U(X_T(\xi^\delta, t+1, x + \alpha \cdot R_{t+1}))|R_{t+1}] + \delta.$$

(b) Give an expression for $v(t, x)$ in the case of exponential utility, i.e.,

$$U(x) = 1 - e^{-\lambda x}$$

for some $\lambda > 0$.

Exercise 6.2 Consider the model studied in Exercises 5.1 and 5.2, i.e., a trinomial model with 1 asset and $m = r = 0$ and $u = -d$.

(a) Use the formula from Corollary 3.25 to find the EMM which minimizes the relative entropy. In other words, calculate P^* according to

$$\frac{dP^*}{dP} = \frac{e^{\lambda^* Y}}{E[e^{\lambda^* Y}]},$$

where λ^* minimizes the moment generating function of the discounted net gains Y.

(b) Use the parametrization from Exercise 1.2 to find the minimizer of the relative entropy using its definition. Verify that this is indeed the probability measure found above.