QUESTIONS FOR THE FINAL EXAMINATION

INTRODUCTION TO MATHEMATICAL FINANCE,
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- These problems are prepared to help you study for the final examination.
- You are expected to know the key definitions and the statements of the key results, their proofs, important examples and how to apply them.
- There will be questions from three categories:
  (1) Definitions and concepts: here you need to know not only the definitions but also their use, importance and relevant examples;
  (2) Proofs (whole or part) of results covered in the lectures: below provides almost all possible questions in this direction;
  (3) Application of the theory in particular questions: problems done in the exercise classes and questions below are good examples.
- Not all the questions will be from the below list.
- You do not need to solve all problems to receive a full grade, but of course a good performance is needed.
- I wish you a nice summer vacation and success in the examination period.
- Corrections in this version are marked red.

I. Definitions and Concepts. In the below questions, a precise definition and few non-trivial examples are expected. You should also be to able discuss the need of such a definition and its use.

(1) Arbitrage. One and multi-step.
(2) Self-financing strategy.
(3) Risk-neutral Measure.
(4) Option, Contingent Claims.
(5) Call or Put Option.
(6) Fair Price of a Contingent Claim.
(7) Super-replication, pricing intervals.
(8) Complete Market.
(9) Binomial, trinomial model.
(10) Black-Scholes price.
(11) Relative interior of a convex set.
(12) Essential Supremum (Lecture Notes).
(13) Rational Preference.
(14) Certainty Equivalent.
(15) Risk aversion and Risk aversion factor.
(16) Arrow-Debrue Equilibrium.

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(17) vonNeumann-Morgenstern Utility.

II. Theorems. In the below questions, you are expected to state the result, and maybe asked to provide part of whole of their proofs.

(1) Fundamental Theorem of Asset Pricing:
one step version (Theorem 1.7),
continent initial data (Theorem 1.55),
multi-step version (Theorem 5.16).
(2) Local to Non-local Arbitrage (Proposition 5.11).
(3) Characterization of no-arbitrage interval:
one-step (Theorem 1.32),
multi-step (Theorem 5.29).
(4) Super-replication theorem:
one-step (Corollary 1.34),
multi-step (Theorem 5.32),
lecture-notes (Theorem 4.2).
(5) Expected Utility and Risk Avresion, (Proposition 2.33).
(6) Utility Maximization and Arbitrage, (Theorem 3.3).
(7) Utility Maximization and Risk Neutral Measure, (Proposition 3.7 and Corollary 3.10).
(8) Exponential Utility and Entropy (Theorem 3.24).
(9) Equilibrium (Example 3.61).

III. Questions. In these class of questions, we expect you to give an almost complete proof. I provide the following list **in addition** to the exercise problems discussed during the semester.

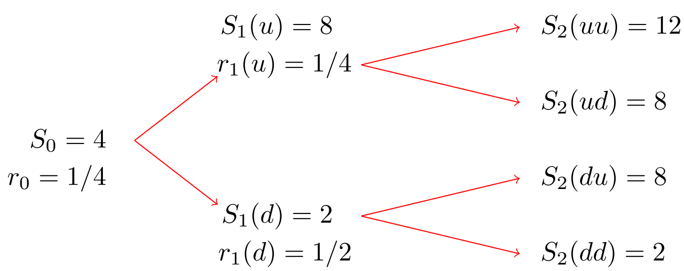
(1) Let $\mathbb{Q}^*$ be a risk-neutral measure and $r > 0$ be the constant interest rate. Set
$$P(K,T) := e^{-rT}\mathbb{E}_{\mathbb{Q}^*}[(K - S_T)^+],$$
for $K, T > 0$. Show that
  a. For fixed $T > 0$, $K \to P(K,T)$ is increasing.
  b. For fixed $T > 0$, $K \to P(K,T)$ is convex.

(2) (This is similar to Exercise 1.3 and 2.5). Let $C(K,T)$ be the price of a European Call Option with maturity T and strike K written on one stock. Assume that interest rate is positive. Show without using any model (in particular, without using risk-neutral pricing) but using the fact that there is no-arbitrage to prove the following statements;
  a. For fixed $T > 0$, $K \to C(K,T)$ is decreasing.
  b. For fixed $T > 0$, $K \to C(K,T)$ is convex.
  c. For fixed $K > 0$, $T \to C(K,T)$ is increasing.

(3) (Problem 4.1, page 115, Shreve's book). Consider a three step Binomial model with $S_0 = 4$ and up factor 2 and a down factor 1/2, i.e, $S_1 \in \{2, 8\}$, $S_2 \in \{1, 4, 16\}$ and $S_3 \in \{0.5, 2, 8, 32\}$. Assume that interest rate $r = 1/4$.
  a. Compute the risk-neutral up probability.
  b. Compute the price V_0^P of the American put that expires at time 3 and has strike 4 (i.e., its pay-off is $g_P(s) = (4 - s)^+$).

c. Compute the price V_0^C of the American call that expires at time 3 and has strike 4 (i.e., its pay-off is $g_C(s) = (s-4)^+$).
d. Compute the price V_0^S of the American straddle that expires at time 3 and has strike 4 (i.e., its pay-off is $g_S(s) = g_P(s) + g_C(s)$).
e. Explain why $V_0^S < V_0^P + V_0^C$.

(4) (Problem 2.9, page 57, Shreve's book). Consider a two step discrete model in which both the stock price and the interest rate evolves in time (Notice that up and down factors also change in time). The filtration is generated by the up-down process.



a. Compute the risk-neutral probabilities and argue that this model this model is complete.
b. Compute the price of the European Call option that expires at time 2 and has strike 7 at all nodes.
c. Determine the replicating portfolio.

(5) Let U be an exponential utility function. Let U be an exponential utility function. Consider the financial market given above and the actual probability measure has equal to probability of up and down (this might be different than the risk neutral probability). Let $\{\xi_1, \xi_2\}$ be a predictable process, i.e, ξ_1 is a constant and ξ_2 may tale two values depending on S_1 is up or down. Solve the utility maximization problem of

$$\max \to \mathbb{E}[U(X_2(\xi))],$$

where $X_2(\xi)$ is the return when under the portfolio ξ,

$$X_2(\xi) = \xi_2 S_2 + (1+r_1)(\xi_1 - \xi_2)S_1 - (1+r_0)(1+r_1)\xi_1 S_0,$$

i.e,

$$\begin{aligned}
X_2(\xi)(uu) &= \xi_2(u)12 + \frac{5}{4}(\xi_1 - \xi_2(u))8 - \frac{25}{16}\xi_1 4, \\
X_2(\xi)(ud) &= \xi_2(u)8 + \frac{5}{4}(\xi_1 - \xi_2(u))8 - \frac{25}{16}\xi_1 4, \\
X_2(\xi)(du) &= \xi_2(d)8 + \frac{3}{2}(\xi_1 - \xi_2(d))2 - \frac{15}{8}\xi_1 4, \\
X_2(\xi)(uu) &= \xi_2(d)2 + \frac{3}{2}(\xi_1 - \xi_2(d))2 - \frac{15}{8}\xi_1 4.
\end{aligned}$$

(6) Let U be an exponential utility function. Suppose the financial market consists of one stock and fixed interest rate $r = 0$ and there are T possible trading times. Hence, we are given a stochastic process $\{S_t\}_{t=0,1,\ldots,T}$. Let C

be a $\mathcal{F}_T$ measurable, bounded random variable (or equivalently a contingent claim). Assume that C achievable and its price is given by

$$p^C := \text{fair price of } \ C = \mathbb{E}_{\mathbb{Q}^*}[C],$$

where $\mathbb{Q}^*$ is a risk neutral measure.

Let $\mathcal{A}$ be the set of all predictable processes (or equivalently portfolios) $\xi = \{\xi_t\}_{t=1,\ldots,T}$ and set

$$G_t(\xi) := \sum_{t=1}^{T} \xi_t[S_t - S_{t-1}].$$

Since C is attainable, there is $\xi^C \in \mathcal{A}$ such that

$$C = p^C + G_T(\xi^C) = p^C + \sum_{t=1}^{T} \xi_t^C[S_t - S_{t-1}].$$

Consider two maximization problems with a given constant $x \in \mathbb{R}$,

$$\begin{aligned} V_f(x) &:= \max_{\xi\in\mathcal{A}} \mathbb{E}_{\mathbb{Q}}[U(x + G_T(\xi))], \\ V_C(x) &:= \max_{\xi\in\mathcal{A}} \mathbb{E}_{\mathbb{Q}}[U(x + G_T(\xi) - C)]. \end{aligned}$$

Given a wealth level x, the *utility indifference price* of C, $p_0(x)$ is defined as the solution of the equation

$$V_f(x) = V_C(x + p_0(x)).$$

Show that p^C is the unique solution of the above equation for every x.

Is the exponential assumption needed? Find the least amount of assumptions needed to make the result correct.