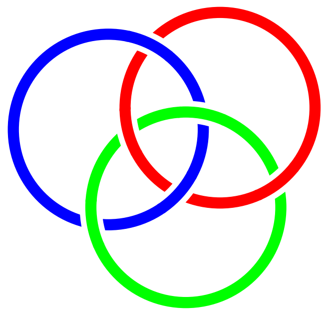
Exercisesheet 3: Simple knot invariants

1. Prove that the unknotting number $u(K)$ is finite.
2. (a) Prove that for any oriented link L the linking number $lk(L)$ is an integer.
   (b) Choose an orientation for the following link, the *Borromean rings*, and compute its linking number. How does the linking number change if you reverse one, two or all three of the components?



3. Prove that $c(K)$ and $u(K)$ are subadditive invariants i.e.
$$c(K_1 \# K_2) \leq c(K_1) + c(K_2)$$
and likewise for $u(K)$.

   Remark: It is conjectured that both these invariants are in fact additive, i.e. that equality holds above, but no one was able to prove that so far.

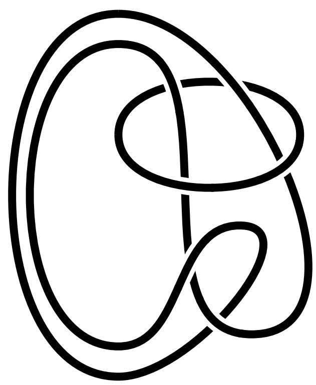
4. Prove that the linking number $lk(L)$ is invariant under Reidemeister move 3.
5. Draw a picture to show that there exist oriented 2-component links with linking number n, for any integer n.
6. Prove that if the orientation on one component of a 2-component oriented link L is reversed then its linking number is negated. Moreover, determine the linking number of the mirror image of L.
7. Prove that any knot diagram has at least three 3-colourings and hence $\tau_3(D) \geq 3$.
8. Show that $\tau_3(L)$ is invariant under Reidemeister move 3.
9. Compute τ_3 of the unknot and $\tau_3(5_1)$ – what do you observe? Can we use τ_3 to distinguish the two knots with crossing number 5?
10. Show that the linking number fails to distinguish the Whitehead link from the unlink, but that τ_3 succeeds.

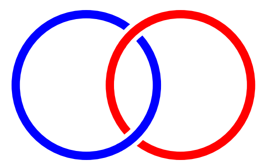
11. (a) Show that $\tau_3(K)$ is always divisible by 3.
    (b) Show that $\tau_3(K)$ is always a power of 3.
    (c) Show that $\tau_3(K) - 3$ is always divisible by 3!.

12. Show that τ_3 cannot distinguish a knot from its mirror image.

13. Use $\tau_3(K_1 \# K_2) = \frac{1}{3}\tau_3(K_1)\tau_3(K_2)$ to prove that there are infinitely many distinct knots.

14. Calculate $\tau_3(L)$ for the following two two-component links. Can you decide whether these links are equivalent or not?





15. Determine for each of the following statements whether it is true or false.

    (a) If K_1 and K_2 are knots with $\tau_3(K_1) = \tau_3(K_2)$ then K_1 is equivalent to K_2.
    (b) If K_1 and K_2 are equivalent knots, then $\tau_3(K_1) = \tau_3(K_2)$.
    (c) If an oriented link has $lk(L) = 0$, then L is equivalent to the unlink.
    (d) There is no knot with $\tau_3(K) = 6$.
    (e) For any knot $\tau_3(\overline{K}) = -\tau_3(K)$.
    (f) The function f defined on knot diagrams by setting

    $$f(D) = \text{number of crossings on D}$$

    defines an invariant of knots.
    (g) For any n, the number of knots with crossing number at most n is finite.
    (h) For any n, the number of knots with unknotting number at most n is finite.
    (i) For any oriented knot K, $\tau_5(rK) = \tau_5(K)$.

(j) For any knot $\tau_2(K) = 2$.

16. Show that the equation that needs to be satisfied for 3-colourability is equivalent to the general equation for p-colourability.

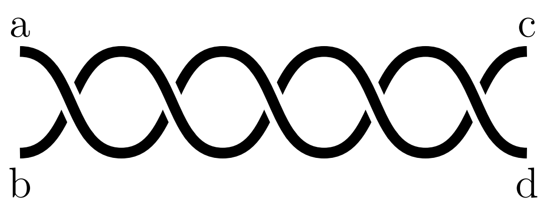
17. Show that τ_2 equals 2 to the power of the number of components of the link and is therefore not very interesting.

18. Let τ_5 be the number of 5-colourings of a link.

    (a) Use τ_5 to show that the figure-eight knot 4_1 is non-trivial.

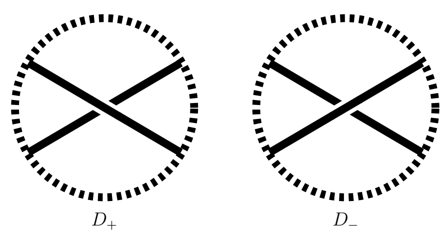
    (b) Calculate $\tau_5(4_1\#4_1)$

    (c) Suppose the diagram below allows non-trivial 5-colourings with the colours a, b, c and d emerging at the ends as shown. Show that $a = c$ and $b = d$.



19. Calculate the number of 5- and 7-colourings of the knot 7_1.

20. Suppose K_+ and K_- are two knots having diagrams D_+ and D_-, which are identical except at one crossing, as shown below.



    (a) Prove that $\tau_p(K_+)$ and $\tau_p(K_-)$ are either equal or one is p times the other.

    (b) Deduce that the unknotting number of a knot that satisfies

$$u(K) \geq \log_p(\tau_p(K)) - 1$$

Due Date: 24.03.2015