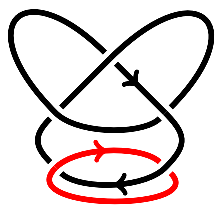
Examplesheet 5: Alternating knots

1. (Jones polynomials revisited)
   (a) Suppose L_+, L_- and L_0 are links differing just at one crossing, as in the skein relation, and that L_+ has μ components. What are the possibilities for the number of components of L_- and L_0?
   (b) Show that for links with an odd number of components (including knots) the Jones polynomial contains only integral powers of t and t^{-1}, and for links with an even number of components it contains only half-integral powers, i.e. ...,$t^{-\frac{3}{2}}$, $t^{-\frac{1}{2}}$, $t^{\frac{1}{2}}$, $t^{\frac{3}{2}}$,... (Hint: Use induction again. Do you think it is possible to prove this result by using only the Kauffman bracket state-sum, not the Jones skein relation?)
2. Show that every alternating knot has a knot diagram which is not alternating.
3. **Open question:** Give an intrinsically 3-dimensional definition of an alternating knot (i.e. do not mention the word knot diagram).
4. Prove that the span or breath of the bracket polynomial $B(< K >)$ is a knot invariant (although the bracket polynomial itself is not).
5. Show that the crossing number of an alternating knot is equal to the span of its Jones polynomial.
6. Prove that
$$c(K_1 \# K_2) = c(K_1) + c(K_2)$$
   if K_1 and K_2 are alternating knots.
   Note: It is unknown whether this equality us true in general (one direction of the inequalities is obvious!).
7. Give an example of an n-crossing diagram D for which $B(< D >) = 0$.
8. Given a knot K which has a reduced alternating diagram with n crossings for n an odd number.
   (a) Show that K is not equivalent to its mirror image $\overline{K}$.
   (b) Can $K \# K$ be equivalent to its mirror image?
9. Prove that the knots 8_{19}, 8_{20} and 8_{21} have no alternating diagram.

More about polynomial invariants

10. Calculate the HOMFLY-polynomial of the Hopf-link where both rings carry opposite orientations.

11. Determine the HOMFLY-polynomial of the trefoil.
12. Show that the HOMFLY-polynomial is a knot invariant (i.e. does not depend on the orientation of the knot).
13. Prove the following properties of the HOMFLY-polynomial:
    (a) $P(L \sqcup O) = -(l + l^{-1})m^{-1}P(L)$
    (b) $P(L_1 \sqcup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2)$
    (c) $P(L_1 \# L_2) = P(L_1)P(L_2)$
14. Use the property for the connected sum of the links L_1 given by



    and $L_2 = 4_1$, i.e. the figure-eight knot, to show that the HOMFLY-polynomial is not a complete invariant for oriented links.
15. Explain what it means that the HOMFLY-polynomial determines the Jones-polynomial by setting $l = it^{-1}$ and $m = i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$.
16. Prove the last part of the proposition about the Conway-polynomial $\nabla_L(z)$ which states that: If $\#L_+ = \#L_- = 1$ then

$$a_2(L_+) - a_2(L_-) = \mathrm{lk}(L_0)$$

17. Determine $\nabla_{3_1}(z)$ and $\nabla_{\overline{3_1}}(z)$ – what do you observe?

Due Date: April 28, 2015