Exercise Sheet 1

Exercise 1

Prove that the Lorentz product $\langle x, y\rangle_{\mathrm{L}} = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$ on R^{n+1} induces a Riemannian metric on $H^n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle_{\mathrm{L}} = 1,\, x_{n+1} > 0\}$.

Exercise 2

(a) Let G be a group, $H < G$ a subgroup. Show that G acts effectively on G/H (that is, e is the only element of G leaving every gH fixed) if and only if H contains no normal subgroup of G other than $\{e\}$.

(b) Suppose that M is a topological space and G is a subgroup of the homeomorphism group of M that acts transitively on M. Show that the stabilizer G_p of any point $p \in M$ contains no normal subgroup of G other than $\{e\}$.

Exercise 3

Prove (using transvections) that every geodesic $\gamma\colon \mathbb{R} \to M$ in a symmetric space M is either injective or periodic.

Exercise 4

Prove that for any pair of points p, q in real hyperbolic n-space H^n and for any orthornormal bases $\{v_i\}$ of TH^n_p and $\{w_i\}$ of TH^n_q there exists an isometry f of H^n such that $Df_p(v_i) = w_i$, for $i = 1, \ldots, n$. (Use the hyperboloid model, where $Isom(H^n) = O(n, 1)_+$.)