Exercise Sheet 2

Exercise 1

1. Let $O(p,q)$ be the semiorthogonal group of all matrices in $GL(p+q,\mathbb{R})$ that preserve the product $\langle x,y\rangle = x_1y_1 + \cdots + x_py_p - x_{p+1}y_p + 1 - \cdots - x_{p+q}y_{p+q}$. Determine the Lie algebra $\mathfrak{o}(p,q)$ (in matrix form) and its Cartan involution and decomposition for the symmetric space

$$G^*_{pq} = O(p,q)/(O(p)\times O(q)) = O(p,q)^0/(SO(p)\times SO(q))$$

(the non-compact Grassmann manifold), in analogy to the case $q=1$ (the hyperbolic space H^p).

Exercise 2

Show that the Killing form B of $\mathfrak{sl}(n,\mathbb{R})$ satisfies

$$B(X,Y) = 2n\ \mathrm{tr}(XY)$$

for all $X,Y\in\mathfrak{sl}(n,\mathbb{R})$.

Exercise 3

Let H be a compact Lie group, and let K be the diagonal in $G := H\times H$. As shown in class, G/K is diffeomophic to H via the map $(g,h)K\mapsto gh^{-1}$. Show that a Riemannian metric on G/K is G-invariant if and only if the corresponding Riemannian metric on H is bi-invariant (that is, both left- and right-invariant).

Prove directly that every compact Lie group H admits a bi-invariant Riemannian metric.

Exercise 4

Let $\mathfrak{g}$ be a (finite-dimensional, real) Lie algebra with Killing form B, and let $\mathfrak{h}$ be any ideal in $\mathfrak{g}$.

(a) Show that $\mathfrak{h}^\perp := \{X\in\mathfrak{g} : B(X,Y) = 0 \text{ for all } Y\in\mathfrak{h}\}$ is an ideal in $\mathfrak{g}$.

(b) Show that if $\mathfrak{g}$ is semisimple, then $\mathfrak{h}\cap\mathfrak{h}^\perp = \{0\}$ and $\mathfrak{g} = \mathfrak{h}+\mathfrak{h}^\perp$.