Exercise Sheet 3

Exercise 1

Let (g, θ) be a symmetric Lie algebra with canonical decomposition $g = k \oplus p$, and let $Z(p) := \{X \in g : [X, Y] = 0 \text{ for all } Y \in p\}$ be the centralizer of p in g.

(a) Show that $Z(p)$ is an ideal in g preserved by θ and that $Z(p) \cap k$ is the greatest ideal of g contained in k. Hence, (g, θ) is effective if and only if the representation $X \mapsto ad_X|_p$ of k on p is faithful.

(b) Suppose now that the symmetric Lie algebra (g, θ) is orthogonal and effective. Show that $z(p)$ is an abelian ideal of g contained in p and that $z(p) = 0$ if and only if g is semisimple.

[Cf. Borel, ch. II, 1.2 and 1.5(c), or Wolf, ch. 8, 8.2.2.]

Exercise 2

Prove that the complexification of $su(n)$ is isomorphic to $sl(n, \mathbb{C})$.

Exercise 3

Check the relation $[p, p] = k$ (Lemma 1.17 in class) explicitly for the sphere $S^2 = SO(3)/SO(2)$. Find a geometric interpretation of this relation.

[Compare Exercise 8 on p. 251 in Helgason.]

Exercise 4

Let (G, K) be a Riemannian symmetric pair and let $\langle \cdot, \cdot \rangle$ be an $Ad(K)$-invariant scalar product on $p \subset g$. Let K^0 be the identity component of K. Show that (G, K_0) is a Riemannian symmetric pair and that the natural map $G/K_0 \to G/K$ of the resulting symmetric spaces is a Riemannian covering.