Exercise Sheet 4

Exercise 1

Let $F: V^4 \to \mathbb{R}$ be a 4-linear function on a vector space V with the same symmetry properties as $(X,Y,Z,W) \mapsto \langle [[X,Y],Z],W\rangle$, for a symmetric bilinear form $\langle \cdot,\cdot\rangle$ and a Lie bracket$[\cdot,\cdot]$. Show that if $F(X,Y,X,Y) = 0$ for all $X, Y \in V$, then $F \equiv 0$.

[We used this in the proof of Theorem 1.25. Compare Helgason, Ch. I, Lemma 12.4.]

Exercise 2

Let $\mathfrak{g}$ be Lie algebra (finite-dimensional, over $\mathbb{R}$), and let $s \subset g$ be a Lie triple system. Show that $[s,s]$ and $[s,s] + s$ are subalgebras of g.

Exercise 3

Determine the flats in $G^*_{pq} = O(p,q)^0/(SO(p) \times SO(q))$ and show that the rank of G^*_{pq} equals $\min\{p,q\}$.

[Compare Exercise 1, Sheet 2.]

Exercise 4

(Iwasawa decomposition of $GL(n,R)$) Show that every $g \in GL(n,R)$ has a unique decomposition $g = kan$ where $k \in O(n)$, a is a diagonal matrix with positive entries, and n is upper triangular with 1's on the diagonal.