Probabilities and statistics

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Q1. Let X and Y two independent standard normal random variables ($N(0,1)$). Define the random variable

$$Z := \begin{cases} X & \text{if } Y \geq 0, \\ -X & \text{if } Y < 0. \end{cases}$$

(a) Compute the distribution of Z.

(b) Compute the correlation between X and Z.

(c) Compute $\mathbb{P}(X + Z = 0)$.

(d) Does (X, Z) follow a multivariate normal distribution?.

Solution:

(a) We just have to compute:

$$\begin{aligned} \mathbb{P}(Z \geq t) &= \mathbb{P}((X \geq t, Y > 0) \vee (X \leq -t, Y \leq 0) \\ &= \frac{1}{2}\mathbb{P}(X \geq t) + \frac{1}{2}\mathbb{P}(X \leq -t) \\ &= \mathbb{P}(X \geq t). \end{aligned}$$

So Z has the same law as X, thus $Z \sim N(0,1)$.

(b) Using the definition of covariance

$$\begin{aligned} Covar(X, Z) &= \mathbb{E}\,(XZ) \\ &= \mathbb{E}\left(X^2 \mathbf{1}_{\{y \geq 0\}}\right) + \mathbb{E}\left(-X^2 \mathbf{1}_{\{y<0\}}\right) = 0. \end{aligned}$$

(c) We have that

$$\mathbb{P}(X + Z = 0) = \mathbb{P}(Y < 0) + \mathbb{P}(Y \geq 0, 2X = 0) = \frac{1}{2}.$$

(d) It's not a multivariate normal, because the sum of them is not a normal.

Q2. **(a)** Take X a random variable. Prove that for all $\lambda \geq 0$

$$\mathbb{P}(X \geq t) \leq \exp(e^{-\lambda t})\mathbb{E}\left(e^{\lambda X}\right).$$

(b) Define $\phi_X(\lambda) := \ln\left(\mathbb{E}\left(e^{\lambda X}\right)\right)$. Prove that $\phi(\lambda) \geq \lambda \mathbb{E}\,(X)$.

(c) Prove that

$$\mathbb{P}(X \geq t) \leq \mathbb{E}\left(e^{-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}}\right).$$

(d) If $X \sim N(0, \sigma)$, calculate $\phi_X(\lambda)$.

(e) Prove that if X is a positive random variable

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t) dt$$

(f) Show that if $X \sim N(0, \sigma)$ and Y is a random variable such that $\psi_Y(\lambda) \leq \psi_X(\lambda)$, then

$$\mathbb{E}(Y^2) \leq 2\sigma^2.$$

Solution:

(a) For $\lambda = 0$ the property is trivial. If $\lambda > 0$, using Markov (chebyshev) inequality we have that

$$\begin{aligned}\mathbb{P}(X \geq t) &= \mathbb{P}(\exp(\lambda X) \geq \exp(\lambda t)) \\ &\leq \frac{1}{\exp(\lambda t)} \mathbb{E}(\exp(\lambda X)).\end{aligned}$$

(b) Given that $\exp(\lambda \cdot)$ is a convex function, thanks to Jensen inequality (Thm 3.6) we have that:

$$\begin{aligned}&\mathbb{E}(\exp(\lambda X)) \geq \exp(\mathbb{E}(\lambda X)) \\ \Rightarrow &\phi(\lambda) = \ln(\mathbb{E}(\exp(\lambda X))) \geq \lambda \mathbb{E}(X).\end{aligned}$$

(c) Using part a) we have that for all $\lambda > 0$

$$\begin{aligned}&\mathbb{P}(X \geq \lambda) \leq \exp(-\lambda t)\mathbb{E}\left(e^{\lambda X}\right) = \exp(-(\phi_X(\lambda) - \lambda t)) \\ \Rightarrow &\mathbb{P}(X \geq \lambda) \leq \exp(e^{-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}}),\end{aligned}$$

where we have just taken Infimum over $\lambda \geq 0$.

(d) Let's compute

$$\begin{aligned}\mathbb{E}\left(e^{\lambda X}\right) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\lambda x} e^{\frac{-x^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\lambda\sigma^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\lambda x} e^{-\frac{(x-\lambda\sigma^2)^2}{2\sigma^2}} e^{\frac{\lambda^2\sigma^2}{2}} dx \\ &= e^{\frac{\lambda^2\sigma^2}{2}},\end{aligned}$$

then $\phi(\lambda) = \frac{1}{2}\lambda^2\sigma^2$.

(e) We just have to use Fubini's Theorem to compute

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{t \leq X\}} dt\right) \\ &= \int_0^\infty \mathbb{E}\left(\mathbf{1}_{\{X \geq t\}}\right) dt \\ &= \int_0^\infty \mathbb{P}(X \geq t) dt.\end{aligned}$$

(f) First let's compute:

$$\sup_{\lambda>0}\left\{\lambda t-\frac{1}{2}\lambda^2\sigma^2\right\},$$

this is just a quadratic function on λ, so it attains it maximum in $\lambda_{\max}=\frac{t}{\sigma^2}>0$. Then its supremum is just $\frac{t}{2\sigma^2}$. So thanks to problem c) we have that

$$\begin{aligned}\mathbb{P}(Y\geq t)&\leq \exp(e^{-\sup_{\lambda\geq 0}\{\lambda t-\phi_Y(\lambda)\}})\\&\leq \exp(e^{-\sup_{\lambda\geq 0}\{\lambda t-\phi_X(\lambda)\}})\\&=\exp\left(-\frac{t^2}{2\sigma^2}\right).\end{aligned}$$

Thus, thanks to part e) we have that

$$\begin{aligned}\mathbb{E}\left(Y^2\right)&=\int_0^\infty P(Y^2\geq t)dt\\&\leq\int_0^\infty \exp\left(-\frac{t}{2\sigma^2}\right)dt\\&=2\sigma^2.\end{aligned}$$

Q3. Let X and Y be random variables with joint density distribution given by

$$f_{X,Y}(x,y)=e^{-x^2y}\mathbf{1}_{\{x\geq 1\}}\mathbf{1}_{\{y\geq 0\}}$$

(a) Why is this a probability measure?

(b) What is the density function of X.

(c) Compute $\mathbb{P}(Y\leq\frac{1}{X^2})$.

Solution:

(a) We just have to prove that $\int_{\mathbb{R}^2}f_{X,Y}(x,y)dxdy=1$.

$$\begin{aligned}\int_1^\infty\int_0^\infty e^{-x^2y}dydx&=\int_1^\infty\left.\left(-\frac{e^{-x^2y}}{x^2}\right)\right|_0^\infty dx\\&=\int_1^\infty\frac{1}{x^2}dx=1.\end{aligned}$$

(b) We now that $f_X(x)=\int_{\mathbb{R}}f_{X,Y}(x,y)dy$, then

$$f_X(x)=\int_0^\infty e^{-x^2y}dy\mathbf{1}_{\{x\geq 1\}}=\frac{1}{x^2}\mathbf{1}_{\{x\geq 1\}}.$$

(c) Let's compute:

$$\begin{aligned}\mathbb{P}\left(Y\leq\frac{1}{X^2}\right)&=\int_1^\infty\int_0^{\frac{1}{x^2}}e^{-x^2y}dxdy\\&=\int_1^\infty\frac{1}{x^2}-e^{-1}\frac{1}{x^2}=1-e^{-1}.\end{aligned}$$

Q4. **(a)** Let μ_n and ν_n two sequence of probability measure on $\mathbb{R}$. and $\epsilon_n \in (0,1)$ with $\epsilon_n \to 0$. Prove that if $\mu_n \to \mu$ in distribution, then $(1-\epsilon_n)\mu_n + \epsilon_n \nu_n \to \mu$ in distribution.

(b) Construct with the help of a) a sequence μ_n so that $\mu_n \to \mu$ in distribution but $\lim_{n\to\infty} \int |x| d\mu_n(x) \neq \int |x| d\mu(x)$.

(c) Prove that if $\mu_n \to \mu$ and $\sup_n \int x^2 d\mu_n(x) = K < \infty$ then

$$\int |x| d\mu_n(x) \to \int |x| d\mu(x).$$

Hint: For all M prove that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} \leq \frac{x^2}{M} d\mu(x).$$

Solution:

(a) Take $f : \mathbb{R} \to \mathbb{R}$ a continuous and bounded function

$$\begin{aligned}\left| \int f d((1-\epsilon_n)\mu_n + \epsilon_n \nu_n) - \int f d\mu \right| &\leq \left| \int f d\mu_n - \int f d\mu \right| + \epsilon_n \left| \int f d\nu_n - \int f d\mu_n \right| \\ &\leq \left| \int f d\mu_n - \int f d\mu \right| + 2\epsilon_n \|f\|_\infty \to 0.\end{aligned}$$

(b) Take $\mu_n = \delta_0$, i.e. $\mu(A) = \mathbf{1}_{\{0 \in A\}}$ and $\nu_n = \delta_n$. It's clear that $\mu_n \to \delta_0$ (it's a constant sequence), so $\left(1 - \frac{1}{n}\right)\mu_n + \frac{1}{n}\nu_n \to \delta_0$, but:

$$\int |x| d\left(\left(1 - \frac{1}{n}\right)\mu_n + \frac{1}{n}\nu_n\right)(x) = \frac{1}{n} n = 1 \neq 0 = \int |x| d\delta_0(x).$$

(c) We know that $\min\{|\cdot|, M\}$ is a bounded continuous function. So it's clear that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} d\mu(x),$$

and thanks to the monotone convergence theorem

$$\int \min\{|x|, M\} d\mu(x) \overset{M\to\infty}{\nearrow} \int |x| d\mu(x)$$

Note that thanks to Cauchy-Schwarz inequality:

$$\begin{aligned}\int |x| \mathbf{1}_{\{|x| \geq M\}} d\mu_n(x) &\leq \sqrt{\int x^2 d\mu_n} \sqrt{\int \mathbf{1}_{\{|x| \geq M\}} d\mu_n(x)} \\ &= \sqrt{K} \sqrt{\int |x| \frac{1}{|x|} \mathbf{1}_{\{|x| \geq M\}} d\mu_n(x)} \\ &\leq K^{\frac{3}{4}} \left(\int \int \frac{1}{|x|} \mathbf{1}_{\{|x| \geq M\}} d\mu_n(x) \right)^{\frac{1}{4}} \\ &\leq K^{\frac{3}{4}} M^{-\frac{1}{4}},\end{aligned}$$

and using the same method

$$\begin{aligned}\int M\mathbf{1}_{\{|x|\geq M\}} d\mu_n(x) &= \int \frac{M}{x} x \mathbf{1}_{\{|x|\geq M\}} d\mu_n(x) \\ &\leq \sqrt{\int x^2 d\mu_n} \sqrt{\int \mathbf{1}_{\{|x|\geq M\}} d\mu_n(x)} \\ &\leq K^{\frac{3}{4}} M^{-\frac{1}{4}}.\end{aligned}$$

To finish, take $\epsilon > 0$, and M so that $K^{\frac{3}{4}} M^{-\frac{1}{4}} < \frac{\epsilon}{4}$ and

$$\left| \int \min\{|x|, M\} d\mu(x) - \int |x| d\mu(x) \right| \leq \frac{\epsilon}{4}.$$

Then, take n_0 so that for all $n \geq n_0$

$$\left| \int \min\{|x|, M\} d\mu_n(x) - \int \min\{|x|, M\} d\mu(x) \right| \leq \frac{\epsilon}{4}.$$

Then

$$\begin{aligned}&\left| \int |x| d\mu_n(x) - \int |x| d\mu(x) \right| \\ \leq &\left| \int |x| d\mu_n(x) - \int \min\{|x|, M\} d\mu_n(x) \right| + \left| \int \min\{|x|, M\} d\mu_n(x) - \int \min\{|x|, M\} d\mu(x) \right| \\ &+ \left| \int \min\{|x|, M\} d\mu(x) - \int |x| d\mu(x) \right| \\ = &\left| \int |x| - M\mathbf{1}_{\{x\geq M\}} d\mu_n(x) \right| + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$