Probabilities and statistics

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Serie 3

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Q1. THE BIRTHDAY PARADOX Take an urn with N balls numerated from $\{1,..,N\}$. Perform the experiment of extracting balls with replacement.

(a) Let $A_n :=$ "The first n balls extracted are different". Calculate $\mathbb{P}(A_n)$ (use a Laplace model).

(b) Prove the following inequalities:

$$1-\frac{n(n-1)}{2N} \le \mathbb{P}(A_n) \le \exp\left(-\frac{n(n-1)}{2N}\right).$$

(c) Calculate $\inf\{n \in \mathbb{N} : \mathbb{P}(A_n) < \frac{1}{2}\}$ for $N = 365$. Relate this problem with the Birthday Problem: " Find the probability that, in a group of N people, there is at least one pair who have the same birthday".

Q2. RANDOM WALK IN $\mathbb{Z}^d$ We are interested in studying the average numbers of visit to 0 of a random walk in $\mathbb{Z}^d$, for this we will need the following definitions:

$$\begin{aligned}
&\Omega := \left(\{-1,1\}^d\right)^N = \{((\omega_{i,j})_{i=1,\dots,d})_{j=1,..N}\},\\
&\mathcal{A} := \{A : A \subseteq \Omega\},\\
&\mathbb{P} = \text{Laplace Model in } \Omega,\\
&S_n = (S_n^{(1)}, \dots, S_n^{(d)}),\\
&S_0 = (0, \dots, 0),\\
&S_n = S_{n-1} + \omega_n,\\
&K_n = |\{0 \le k \le n : S_k = (0,0,\dots.,0)\}|.
\end{aligned}$$

(a) Why can we think of S_n as a Random Walk in $\mathbb{Z}^d$?. What will be the value of $(S_n)_{n=0}^5$ if $d = 2$, $N = 5$ and

$$\omega = ((1,-1),(-1,-1),(1,1),(1,1),(-1,-1)).$$

(b) Compute $\mathbb{P}\left(S_{2n} = (0,0,\dots,0)\right)$. Use the Stirling Formula (Skript (2.2.30)) to characterize the limiting behavior.

(c) Write K_n as a sum of indicators functions. From which dimension d

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[K_n\right] < \infty?.$$

Q3. GAMBLER'S RUIN Peter and John are playing the following game. Each minute they flip a coin. If the coin is head Peter gives one Frank to John, and if it is tail John gives one Frank to Peter. At the beginning of the game Peter has F Franks and John have H Franks. The games stops only if either Peter or John have no more money. Define P_n (J_n resp.) as the probability that Peter (John resp.) has already won on time n.

(a) Prove that $P_n + J_n \nearrow 1$.
Hint: Define $S_k :=$ "The money that Peter has won in the game" and try to see the problem as a random walk problem.

(b) Calculate $P = \lim P_n$.
Hint:. Take the stopping time

$$T_n := \min\{n, \min\{k > 0 : S_k \in \{-F, H\}\}\},$$

and use the Stopping Time Theorem (2.2.5. in the Skript).

(c) Now, suppose that the game will stop only when John is in bankruptcy (i.e. when he has no more money). Why is this equivalent to $F = \infty$ in the previous set?. Prove that $P_n \nearrow 1$.

(d) Define

$$\begin{aligned} T_n &:= \inf\{n, \min\{k > 0 : \text{John has 0 Franks at time } k\}\}, \\ T &:= \inf\{k > 0 : \text{John has 0 Franks at time } k\}. \end{aligned}$$

Prove that for all $\omega \in \Omega$, $T_n(\omega) \nearrow T(\omega)$ and that $\mathbb{P}(H - S_{T_n} = 0) \nearrow 1$. Why doesn't this contradict the Stopping Time Theorem (2.2.5).
Hint: Note that we could define $S_T = \lim S_{T_n}$ and $\mathbb{E}[S_T] \neq 0$.

Have a nice week ♫!!.