Probabilities and statistics
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Serie 8

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Q1. Let X and Y two independent standard normal random variables ($N(0,1)$). Define the random variable

$$Z := \begin{cases} X & \text{if } Y \geq 0, \\ -X & \text{if } Y < 0. \end{cases}$$

(a) Compute the distribution of Z.

(b) Compute the correlation between X and Z.

(c) Compute $\mathbb{P}(X + Z = 0)$.

(d) Does (X, Z) follow a multivariate normal distribution?.

Q2. **(a)** Take X a random variable. Prove that for all $\lambda \geq 0$

$$\mathbb{P}(X \geq t) \leq \exp(e^{-\lambda t})\mathbb{E}\left(e^{\lambda X}\right).$$

(b) Define $\phi_X(\lambda) := \ln\left(\mathbb{E}\left(e^{\lambda X}\right)\right)$. Prove that $\phi(\lambda) \geq \lambda \mathbb{E}(X)$.

(c) Prove that

$$\mathbb{P}(X \geq t) \leq \mathbb{E}\left(e^{-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}}\right).$$

(d) If $X \sim N(0, \sigma)$, calculate $\phi_X(\lambda)$.

(e) Prove that if X is a positive random variable

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t)dt$$

(f) Show that if $X \sim N(0, \sigma)$ and Y is a random variable such that $\psi_Y(\lambda) \leq \psi_X(\lambda)$, then

$$\mathbb{E}(Y^2) \leq 2\sigma^2.$$

Q3. Let X and Y be random variables with joint density distribution given by

$$f_{X,Y}(x,y) = e^{-x^2 y}\mathbf{1}_{\{x \geq 1\}}\mathbf{1}_{\{y \geq 0\}}$$

(a) Why is this a probability measure?

(b) What is the density function of X.

(c) Compute $\mathbb{P}(Y \leq \frac{1}{X^2})$.

Q4. **(a)** Let μ_n and ν_n two sequence of probability measure on $\mathbb{R}$. and $\epsilon_n \in (0,1)$ with $\epsilon_n \to 0$. Prove that if $\mu_n \to \mu$ in distribution, then $(1-\epsilon_n)\mu_n + \epsilon_n \nu_n \to \mu$ in distribution.

(b) Construct with the help of a) a sequence μ_n so that $\mu_n \to \mu$ in distribution but $\lim_{n\to\infty} \int |x| d\mu_n(x) \neq \int |x| d\mu(x)$.

(c) Prove that if $\mu_n \to \mu$ and $\sup_n \int x^2 d\mu_n(x) = K < \infty$ then

$$\int |x| d\mu_n(x) \to \int |x| d\mu(x).$$

HINT: For all M prove that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} \leq \frac{x^2}{M} d\mu(x).$$