Probabilities and statistics
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Serie 1

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Q1. We throw simultaneously two dices, one green and one red. Consider the following events:

- $W_1 :=$ Neither of the dices has a result greater than 2.
- $W_2 :=$ The green and the red one have the same number on them.
- $W_3 :=$ The number on the green is 3 times the number on the red.
- $W_4 :=$ The number on the red is by one greater than the number on the green one.
- $W_5 :=$ The number of the green one is greater or equal than the number on the red one.

(a) Write a suitable space Ω where all of these events can live.

(b) Describe W_i as a subsets of Ω.

(c) If you were colorblind (you cannot differentiate green and red). How does the sample space Ω change?, which W_i can live in this space?.

Solution

(a) $\Omega = \{1,2,3,4,5,6\}^2$. The first coordinate will represent the green dice and the second one the red dice.

(b)
- $W_1 := \{(x,y) \in \Omega : x \leq 2, y \leq 2\} = \{1,2\}^2$.
- $W_2 := \{(x,y) \in \Omega : x = y\} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.
- $W_3 := \{(x,y) \in \Omega : x = 3y\} = \{(3,1),(6,2)\}$.
- $W_4 := \{(x,y) \in \Omega : x+1 = y\} = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$.
- $W_5 := \{(x,y) \in \Omega : x \geq y\}$.

(c) We can define the equivalence relation $\sim$ as

$$(x,y) \sim (z,w) \Leftrightarrow \{x,y\} = \{z,w\},$$

then $\tilde{\Omega} = \Omega/\sim = \{\{x,y\} : x,y \in \Omega\}$. The W_i that will survive are those so that if $(x,y) \in W_i$ then $(y,x) \in W_i$. This happens only for W_1, W_2.

Q2. You have an urn with $4k$ balls each one numbered with a different number in $\{1, ..., 4k\}$. At time j you take out one ball, look at its number and put it back, you repeat this experiment n times. Define

- $A_j :=$ The number taken out in the j-th time is bigger than $2k$.
- $B_j :=$ The number taken out in the j-th time is even.

(a) Write in terms of $(A_j)_{j=1}^n$ and $(B_j)_{j=1}^n$ the following events

i. $A :=$ Between 1 and n there was never a number bigger than $2k$.

ii. $B :=$ Between 1 and n there was at least one even number.

iii. $C :=$ The amount of balls bigger than $2k$ is bigger or equal than the amount of even balls.

(b) Describe in words the following events

i. $\left(\bigcup_{j=1}^{n}(A_j)^c\right)^c$.

ii. $\bigcup_{j=1}^{n-2}(A_j \cap A_{j+1} \cap B_{j+2})$.

iii. $\bigcup_n^n \bigcap_{j=n}^n (A_j \cap B_j)$.

(c) For all $A \subseteq \Omega = \{1, .., 4k\}^n$ define

$$\mathbb{P}(A) = \frac{|A|}{(4k)^n},$$

Show that for all strictly increasing sequences $(j_m)_{m=1}^{N_j}$, $(l_m)_{m=1}^{N_l}$ we have that

$$\mathbb{P}\left(\bigcap_{m=1}^{N_j} A_{j_m} \cap \bigcap_{m=1}^{N_l} B_{l_m}\right) = \left(\frac{1}{2}\right)^{N_l+N_j}.$$

Solution

(a) i. $A = \bigcap_{j=1}^n A_j^c$.

ii. $B = \bigcup_{j=1}^n B_j$.

iii. $C = \bigcup_{\substack{C,D\subseteq\{1,\dots,n\}\\ |C|=|D|}} \left(\bigcap_{j\in C} A_j \cap \bigcap_{j\in D^c} B_j^c\right)$.

(b) i. $\left(\bigcup_{j=1}^{n}(A_j)^c\right)^c = \bigcap_{j=1}^{n} A_j$: All the number taken are bigger than $2k$.

ii. $\bigcup_{j=1}^{N-2}(A_j \cap A_{j+1} \cap B_{j+2})$: There exists one moment where in two consecutive drawings we got a number bigger than $2k$ and in the following extraction we got an even number.

iii. $\bigcup_n^N \bigcap_{j=n}^N A_j \cap B_j$:There is a moment after we only extract number which are even and bigger than $2k$.

(c) Take $B_j \subseteq \{1, ...n\}$. It's easy to prove by induction that for sets of the form

$$\bigotimes_{j=1}^{n} B_j := \{(x_i)_{i=1}^n : x_i \in B_j\},$$

we have that

$$\left| \bigotimes_{j=1}^{n} B_j \right| = \prod_{j=1}^{n} |B_j|.$$

Define

$$A := \{j \in \mathbb{N} : 4k \geq j > 2k\}$$
$$B := \{j \in \mathbb{N} : 4k \geq j = 2\tilde{j}, \tilde{j} \in \mathbb{N}\},$$

It's clear that $A, B \subseteq \{j \in \mathbb{N} : j \leq 4k\}$ and

$$|A| = |B| = 2k = 2|A \cap B|.$$

For simplifying notation in this exercise we will use the following notation. Take $E \subseteq \{1, ..., n\}$ a set and $\epsilon \in \{0, 1\}$ define

$$E^\epsilon = \begin{cases} E & if\ \epsilon = 1, \\ \{1, ..., n\} & if\ \epsilon = 0. \end{cases}$$

Then we just have to realize that:

$$\bigcap_{m=1}^{N_j} A_{j_m} \cap \bigcap_{m=1}^{N_l} B_{l_m} = \bigotimes_{j=1}^{n} \left(A^{\mathbf{1}_{\{j=j_m, m\in\mathbb{N}\}}} \cap B^{\mathbf{1}_{\{j=l_m, m\in\mathbb{N}\}}} \right),$$

then

$$\frac{\left| \left\{ \bigcap_{m=1}^{N_j} A_{j_m} \cap \bigcap_{m=1}^{N_l} B_{l_m} \right\} \right|}{(4k)^n} = \frac{1}{(4k)^n} \bigotimes_{j=1}^{n} (4k) \frac{1}{2}^{\mathbf{1}_{\{j=j_m, m\in\mathbb{N}\}}} \frac{1}{2}^{\mathbf{1}_{\{j=l_m, m\in\mathbb{N}\}}}$$
$$= \left(\frac{1}{2} \right)^{N_l + N_j}.$$

Q3. Let $(A_j)_{j=1}^n$ be events:

(a) Show that:

$$\mathbf{1}_{\bigcup_{j=1}^n A_j} = 1 - \prod_{j=1}^n (1 - \mathbf{1}_{A_j}),$$

use it to prove that:

$$\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathbb{P}\left[\bigcap_{j=1}^k A_{i_j}\right]$$

(b) Using induction prove the following statements:

$$\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] \leq \sum_{j=1}^n \mathbb{P}[A_j] - \sum_{j=1}^{n-1} \mathbb{P}\left[A_j \cap A_{j+1}\right]$$

$$\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] \geq \sum_{j=1}^n \mathbb{P}[A_j] - \sum_{i,j=1, i \neq j}^{n} \mathbb{P}\left[A_j \cap A_{j+1}\right]$$

Solution

(a) It's clear that

$$\begin{aligned}
\mathbf{1}_{\bigcup_{j=1}^n A_j} &= 1 - \mathbf{1}_{\bigcap_{j=1}^n A_j^c} \\
&= 1 - \prod_{j=1}^n \mathbf{1}_{A_j^c} \\
&= 1 - \prod_{j=1}^n \left(1 - \mathbf{1}_{A_j}\right),
\end{aligned}$$

Then we can just use the product formula to have:

$$\prod_{j=1}^n \left(1 - \mathbf{1}_{A_j}\right) = 1 + \sum_{k=1}^n \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^k -\mathbf{1}_{A_{i_k}}.$$

Taking expectations at both sides

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] &= \mathbb{E}\left[\mathbf{1}_{\bigcup_{j=1}^n A_j}\right] \\
&= -\mathbb{E}\left[\sum_{k=1}^n \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^k -\mathbf{1}_{A_{i_k}}\right] \\
&= \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathbb{P}\left[A_{i_1} \cap \ldots \cap A_{i_k}\right].
\end{aligned}$$

(b) We want to use induction. The base case are clearly true, i.e., when $n = 1$ both formulas are true.For the inductive step suppose the formulas are true for an integer n and we will try to prove them for $n+1$. The first one:

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{j=1}^{n+1} A_j\right] &= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j \cup A_{n+1}\right] \\
&= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[\bigcup_{j=1}^{n} A_j \cap A_{n+1}\right] \\
&\leq \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[A_n \cap A_{n+1}\right] \\
&\leq \sum_{j=1}^{n} \mathbb{P}[A_j] - \sum_{j=1}^{n-1} \mathbb{P}\left[A_j \cap A_{j+1}\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[A_n \cap A_{n+1}\right] \\
&= \sum_{j=1}^{n+1} \mathbb{P}[A_j] - \sum_{j=1}^{n} \mathbb{P}\left[A_j \cap A_{j+1}\right].
\end{aligned}$$

For the second formula we have:

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{j=1}^{n+1} A_j\right] &= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j \cup A_{n+1}\right] \\
&= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[\bigcup_{j=1}^{n} (A_j \cap A_{n+1})\right] \\
&\geq \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \sum_{j=1}^{n} P(A_j \cap A_{n+1}) \\
&\geq \sum_{j=1}^{n} \mathbb{P}[A_j] - \sum_{i,j=1,i\neq j}^{n} \mathbb{P}\left[A_j \cap A_i\right] + \mathbb{P}\left[A_{n+1}\right] - \sum_{j=1}^{n} P(A_j \cap A_{n+1}) \\
&= \sum_{j=1}^{n+1} \mathbb{P}[A_j] - \sum_{i,j=1,i\neq j}^{n+1} \mathbb{P}\left[A_j \cap A_i\right].
\end{aligned}$$