Probabilities and statistics

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Let Y be a random variable. We say that Y is infinitely divisible if for all $n \in \mathbb{N}$ there exists $(\xi_i^{(n)})_{i=1}^n$ i.i.d. so that

$$\sum_{i=1}^{n} \xi_i^{(n)} \stackrel{Dist}{=} Y.$$

Q1. We say that a random variable Y is bounded if there exists $M_Y \in \mathbb{R}$ such that $\mathbb{P}(|Y| \leq M_Y) = 1$. We want to understand bounded infinitely divisible random variables.

(a) Prove that if Y is a constant (i.e. there exists $a \in \mathbb{R}$ so that $\mathbb{P}(X = a) = 1$) then Y is a bounded infinitely divisible random variable.

(b) Suppose that Y is bounded and infinitely divisible. Take $\xi_i^{(n)}$ i.i.d. such that $\sum_{i=1}^n \xi_i^{(n)} = Y$, show that $\mathbb{P}(|\xi^{(n)}| \leq \frac{M_Y}{n}) = 1$.

(c) Prove that Y is constant.
Hint: Prove that $Var(Y) = 0$.

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Solution:

(a) Take $a \in \mathbb{R}$ so that $\mathbb{P}(Y = a) = 1$. For $n \in \mathbb{N}$ take $(\xi_i^{(n)})$ such that $\mathbb{P}(\xi_i^{(n)} = \frac{a}{n}) = 1$. Then, it's clear that $\sum_{i=1}^n \xi_i = Y$.

(b) Suppose that $\mathbb{P}(|\xi_1^{(n)}| > \frac{M_y}{n}) \neq 0$, then that means that there exists an $\epsilon > 0$, so that $\mathbb{P}\left(|\xi_1^{(n)}| \geq \frac{M_y}{n} + \epsilon\right) \neq 0$. Without lost of generality $\mathbb{P}(\xi_1^{(n)} \geq \frac{M_y}{n} + \epsilon) = c > 0$. Then we have that

$$\mathbb{P}(Y \geq M_Y + n\epsilon) \geq \mathbb{P}\left(\bigcap_{i=1}^{n} \{\xi_i^{(n)} \geq \frac{M_y}{n} + \epsilon\}\right) = c^n > 0,$$

what is a contradiction.

(c) Note that

$$\begin{aligned} Var\left(\xi_1^{(n)}\right) &= \mathbb{E}\left(\left(\xi_1^{(n)}\right)^2\right) - \mathbb{E}\left(\xi_1^{(n)}\right)^2 \\ &\leq \mathbb{E}\left(\left(\xi_1^{(n)}\right)^2\right) \\ &\leq \frac{M_Y^2}{n^2}, \end{aligned}$$

then, thanks to the independence of $\left(\xi_i^{(n)}\right)_{n\in\mathbb{N}}$

$$Var(Y) = \sum_{i=1}^{n} Var(\xi_i^{(n)}) \leq n\frac{M^2}{n^2} \to 0,$$

so $Var(Y) = 0$. That means that $\mathbb{E}\left((Y - \mathbb{E}(Y))^2\right) = 0$, thus $\mathbb{P}(Y - \mathbb{E}(Y) = 0) = 1$.

Q2. Define $\phi_Y(\lambda) := \mathbb{E}\left(e^{i\lambda Y}\right)$ the characteristic function of Y. We want to understand the characteristic function of infinitely divisible random variables.

(a) Prove that if $Y \sim N(\mu, \sigma^2)$ the Y is infinitely divisible.

(b) Prove that Y is infinitely divisible iff for all n there exists $\phi_{n,Y}$, a characteristic function of a random variable, such that $(\phi_{n,Y}(\lambda))^n = \phi_Y(\lambda)$.

(c) Prove that if Y is infinitely divisible and $\tilde{Y}$ is an independent copy of Y, then $X := Y - \tilde{Y}$ is infinitely divisible. Additionally, show that $0 \le \phi_X(\lambda) \in \mathbb{R}$ for all $\lambda \in \mathbb{R}$.

(d) Prove that $\phi_X(\lambda)^{\frac{1}{n}} = \phi_{n,X}(\lambda)$.
Hint: It may be useful to prove that $0 \le \phi_{n,X}(\lambda) \in \mathbb{R}$.

(e) Prove that for all $\lambda \in \mathbb{R}$, $\phi_{n,X}(\lambda) \to \psi(\lambda)$ a function that is continuous in a neighborhood of 0.

(f) Prove that for all $\lambda \in \mathbb{R}$, $\Phi_X(\lambda) \neq 0$. Conclude that $\Phi_Y(\lambda) \neq 0$.

Solution:

(a) Given $n \in \mathbb{N}$ take $\xi_i^{(n)} \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$. Then, thanks to the independence of $(\xi_i(n))_{i=1}^n$, $\sum_{i=1}^n \xi_i^{(n)} \sim N(\mu, \sigma^2)$.

(b) $\Rightarrow$ If Y is infinitely divisible then for all $n \in \mathbb{N}$ there exists $(\xi_i^{(n)})_{i=1}^n$ i.i.d. So that

$$\sum_{i=1}^n \xi_i^{(n)} \overset{Dist}{=} Y.$$

Thus, thanks to the independence

$$\begin{aligned}
\phi_Y(\lambda) &= \mathbb{E}\left(e^{i\lambda Y}\right) \\
&= \mathbb{E}\left(e^{i\lambda \sum_{i=1}^n \xi_i^{(n)}}\right) \\
&= \prod_{i=1}^n \mathbb{E}\left(e^{i\lambda \xi_1^{(n)}}\right) \\
&= (\phi_{\xi_1^{(n)}}(\lambda))^n.
\end{aligned}$$

$\Leftarrow$ If $\phi_Y(\lambda) = \phi_{n,Y}(\lambda)^n$ and $\phi_{n,Y}$ is the characteristic function of ξ. Take $(\xi_i)_{i=1}^n$, n independent copies of ξ. We have, by the same calculations than before,

$$\phi_{\sum_{i=1}^n \xi_i}(\lambda) = \left(\phi_{\xi_1^{(n)}}(\lambda)\right)^n = \phi_Y.$$

Because their characteristic function are equal we have that $\sum_{i=1}^n \xi_i \overset{dist}{=} Y$.

(c) Take $n \in \mathbb{N}$ and $(\xi_i^{(n)})_{i=1}^n$ i.i.d such that $\sum_{i=1}^n \xi_i^{(n)} = Y$ and $(\tilde{\xi}_i^{(n)})_{i=1}^n$ and independent copy of $(\xi_i^{(n)})_{i=1}^n$, then $X = \sum_{i=1}^n \xi_i^{(n)} - \tilde{\xi}_i^{(n)} \overset{Dist}{=} Y - \tilde{Y}$. So X is infinitely divisible. Then its characteristic function is given by

$$\begin{aligned}
\phi_x(\lambda) &= \mathbb{E}\left(e^{i\lambda(Y-\tilde{Y})}\right) \\
&= \mathbb{E}\left(e^{i\lambda Y}\right)\mathbb{E}\left(e^{-\lambda\tilde{Y}}\right) \\
&= \phi_Y(\lambda)\overline{\phi_Y(\lambda)} = \|\phi_Y(\lambda)\|^2 \ge 0
\end{aligned}$$

(d) We know that $(\phi_{n,X}(\lambda))^n = \phi_X(\lambda)$, then $\phi_{n,X}(\lambda) = \phi_X(\lambda)^{\frac{1}{n}} e^{\frac{2ik\pi}{n}}$. But

$$\phi_{n,X}(\lambda) = \mathbb{E}\left(e^{i\lambda \xi_1^{(n)} - \tilde{\xi}_1^n}\right) = \|\phi_{\xi_1^{(n)}}\|^2 \geq 0,$$

then $\phi_{n,X}(\lambda) = \phi_X(\lambda)^{\frac{1}{n}}$.

(e) Given that $0 \leq \phi_X(\lambda) \leq 1$ we have that $\phi_X(\lambda)^{\frac{1}{n}} \to \mathbf{1}_{\{\phi_X(\lambda)\neq 0\}}$.Given that $\phi_X(0) = 1$ and ϕ_X is continuous, we have that there exists $\epsilon > 0$ so that for all $|\lambda| < \epsilon$ $\mathbf{1}_{\{\phi_X(\lambda)\neq 0\}} = 1$. Then the limit function is continuous in a neighborhood of 0.

(f) Thanks to Lévy's Theorem that ψ is the characteristic function of a random variable ξ. Then we have that $\mathbb{E}(\xi) = -i\psi'(0) = 0$ and that $\mathbb{E}(\xi^2) = \psi''(0) = 0$, so $\mathbb{P}(\xi = 0) = 1$, then

$$\mathbf{1}_{\{\psi_X(\lambda)\neq 0\}} = \psi(\lambda) = \phi_\xi(\lambda) = 1,$$

so $\psi_X(\lambda) \neq 0$ for all $\lambda > 0$. Note that $\psi_X = \|\psi_Y(\lambda)\|^2$, given that the left one is never 0 we have that for all $\lambda \in \mathbb{R}$, $\psi_Y(\lambda) \neq 0$.

Q3. In this question we want to use the criteria of the question one and two to see whether a random variable is infinitely divisible or not.

(a) Let $Y \sim U(0,1)$ is it an infinitely divisible random variable?

(b) Let $(\eta_i)_{i\in\mathbb{N}}$ a sequence of i.i.d random variables. Take $N \sim P(\varsigma)$ independent of $(\eta_i)_{i\in\mathbb{N}}$. Compute, in terms of ϕ_{η_1}, the characteristic function of $Y := \sum_{i=1}^N \eta_i$. Is it an infinitely divisible random variable?
Remember that if $N \sim P(\varsigma)$

$$\mathbb{P}(N = k) = e^{-\varsigma}\frac{\varsigma^k}{k!} \quad k \in \mathbb{N}.$$

Solution:

(a) Given that Y is bounded and it's not constant, it's not an infinitely divisible random variable.

(b) Computing

$$\begin{aligned}
\phi_Y(\lambda) &= \mathbb{E}\left(e^{i\lambda \sum_{i=1}^N \eta_i}\right) \\
&= \sum_{n=0}^\infty \mathbb{E}\left(\mathbf{1}_{\{N=n\}} \exp\left(\lambda i \sum_{k=1}^n \eta_k\right)\right) \\
&= \sum_{n=0}^\infty \mathbb{P}(N=n)\mathbb{E}\left(\exp\left(\lambda i \sum_{k=1}^n \eta_k\right)\right) \\
&= \sum_{n=0}^\infty e^{-\varsigma}\frac{\varsigma^n}{n!}\mathbb{E}\left(\exp\left(i\lambda\eta_1\right)\right)^n \\
&= e^{-\varsigma}\sum_{n=0}^\infty \frac{(\varsigma\phi_{\eta_1}(\lambda))^n}{n!} \\
&= e^{-\varsigma}\exp\left(\varsigma\phi_{\eta_1}(\lambda)\right) \\
&= \exp\left(-\varsigma(1-\phi_{\eta_1}(\lambda))\right).
\end{aligned}$$

It's clear that for all $n \in \mathbb{N}$, we have that $\phi_{Y,n} = \exp\left(-\frac{\varsigma}{n}(1 - \phi_{\eta_1}(\lambda))\right)$, is the characteristic function of $\tilde{Y} := \sum_{i=1}^{\tilde{N}} \eta_i$, where $\tilde{N} \sim P(\frac{\varsigma}{n})$. Then, Y is an infinitely divisible random variable.