Probabilities and statistics
Lecturer: Prof. Dr. Sara van de Geer
Prof. Dr. Martin Larsson

Serie 6

March 30th, 2015

Q1. Memorylessness of exponential random variable. We say that a random variable X has an exponential distribution of parameter λ ($\mathcal{E}(\lambda)$) if for all $t \geq 0$:

$$\mathbb{P}(X \geq t) = e^{-\lambda t}.$$

(a) Find the density function (with respect to the Lebesgue Measure) of an exponential random variable . Calculate its mean and its variance.

(b) Show that if $X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$ and $X_1 \perp X_2$, then $\min\{X_1, X_2\} \sim \mathcal{E}(\lambda_1 + \lambda_2)$.

(c) Show that

$$\mathbb{P}\left(X \geq t + h \mid X \geq h\right) = \mathbb{P}(X \geq t).$$

This property is called memorylessness. We want to prove that the only random variable that has the memorylessness property is the exponential random variable. Suppose that $Y : \Omega \mapsto \mathbb{R}^+$ has the memorylessness property, i.e.,

$$\mathbb{P}\left(Y \geq t + h \mid Y \geq h\right) = \mathbb{P}(X \geq t).$$

(d) Define $G(t) := \mathbb{P}(Y \geq t)$ and prove that $G(t + h) = G(t)G(h)$.

(e) Prove that for all $m, n \in \mathbb{N}$, $G\left(\frac{m}{n}\right) = G(1)^{\frac{m}{n}}$.

(f) Using the monotone property of G prove that for all $t \geq 0$ $G(t) = G(1)^t$. Conclude that Y has an exponential distribution and make explicit the parameter.

Solution

(a) To find the density we just have to derive the CDF

$$F(t) := \mathbb{P}(X \leq t) = 1 - \mathbb{P}(X \geq t) = 1 - e^{-\lambda t}.$$

Then its density is

$$f(t) := F'(t) = \lambda e^{-\lambda t}.$$

We can calculate its mean as

$$\begin{aligned}\mathbb{E}\left(X\right) &= \int_0^\infty t\lambda e^{-\lambda t}dt \\ &= -te^{-\lambda t} \mid_0^\infty + \int_0^\infty e^{-\lambda t}dt \\ &= \frac{1}{\lambda}.\end{aligned}$$

Then its second moment is

$$\begin{aligned}\mathbb{E}\left(X^2\right) &= \int_0^\infty \lambda t^2 e^{-\lambda t} dt \\ &= -t^2 e^{-\lambda t} \mid_0^\infty + 2\int_0^\infty t e^{-\lambda t} dt \\ &= \frac{2}{\lambda^2}.\end{aligned}$$

In conclusion

$$Var(X) = \mathbb{E}\left(X^2\right) - \mathbb{E}\left(X\right)^2 = \frac{1}{\lambda^2}.$$

(b) We just have to compute

$$\begin{aligned}\mathbb{P}\left(\min\{X_1, X_2\} > t\right) &= \mathbb{P}(X_1 > t, X_2 > t) \\ &= \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > t) \\ &= e^{-\lambda_1 t} e^{-\lambda_2 t} \\ &= e^{-(\lambda_1+\lambda_2)t}.\end{aligned}$$

This is the definition of $\min\{X_1, X_2\} \sim \mathcal{E}(\lambda_1 + \lambda_2)$.

(c) We just have to compute

$$\mathbb{P}\left(Y \geq t + h \mid Y \geq h\right) = \frac{\mathbb{P}(Y \geq t+h)}{\mathbb{P}(Y \geq h)} = e^{\lambda t} = \mathbb{P}(Y \geq t).$$

(d) We have to compute

$$\begin{aligned}G(t+h) &= \mathbb{P}(Y \geq t+h) \\ &= \frac{\mathbb{P}(Y \geq t+h)}{\mathbb{P}(Y \geq h)}\mathbb{P}(Y \geq h) \\ &= \mathbb{P}(Y \geq t+h \mid Y \geq h)\mathbb{P}(Y \geq h) \\ &= G(t)G(h).\end{aligned}$$

(e) First we will prove by induction that for all $n \in \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ we have that $G(\sum_{i=1}^n a_i) = \prod_{i=1}^n G(a_i)$. It's clear when $n = 1$, then assuming it's true for n

$$G\left(\sum_{i=1}^{n+1} a_i\right) = G(a_{n+1})G\left(\sum_{i=1}^{n} a_n\right) = \prod_{i=1}^{n+1} G(a_i).$$

Take $m, n \in \mathbb{N}$, we have that

$$\begin{aligned}G(1)^m = G\left(\sum_{i=1}^m 1\right) = G\left(\sum_{i=1}^n \frac{m}{n}\right) = G\left(\frac{m}{n}\right)^n \\ \Rightarrow G(1)^{\frac{m}{n}} = G\left(\frac{m}{n}\right)\end{aligned}$$

(f) Finally, take $t \in \mathbb{R}^+$ and $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ so that $t_n \nearrow t$ and $s_n \searrow t$. Then thanks to the monotonicity of $G(t)$

$$\begin{aligned} &G(t_n) \leq G(t) \leq G(s_n) \\ \Rightarrow &G(1)^{t_n} \leq G(t) \leq G(1)^{s_n} \\ \Rightarrow &G(t) = G(1)^t. \end{aligned}$$

Finally we have that $\mathbb{P}(Y \geq t) = G(1)^t = e^{-\ln(\frac{1}{G(1)})t}$, then $Y \sim \mathcal{E}\left(\ln\left(\frac{1}{G(1)}\right)\right)$.

Q2. Borel Cantelli

(a) Construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a series of measurable sets $(A_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$ and $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 0$.

(b) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Take $(U_n)_{n\in\mathbb{N}}$ a series of uniform independent random variables on $(0,1)$, i.e., for $0 \leq x \leq 1$, $\mathbb{P}(U_n \in [0,x]) = x$.

i. Show that:

$$\mathbb{P}\left((\exists \alpha > 1) \liminf n^\alpha U_n \in \mathbb{R}\right) = 0.$$

Hint: It may be useful to define, for $\alpha > 1$ $A_n^\alpha := \{U_n < n^{-\alpha}\}$. Do not forget that the countable union of sets of probability 0 has probability 0.

ii. Prove that:

$$\mathbb{P}\left(\liminf nU_n \in \mathbb{R}\right) > 0.$$

Solution

(a) Take $([0,1], \mathcal{B}(0,1), \lambda)$ as a probability space and U the identity function. U is distributed as an uniform random variable on $(0,1)$. Define $A_n := \{x \in (0,1) : U(x) \in [0, \frac{1}{n}]\}$. Then we have that $\mathbb{P}(A_n) = \frac{1}{n}$, so $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$. Additionally $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq 0} A_k$ iff $x = 0$, so $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq 0} A_k\right) = 0$.

(b) i. We will use Borel-Cantelli 1) (Skript Lemma 3.1 p. 36). Define $A_n^\alpha := \{U_n < n^{-\alpha}\}$, then:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

so $\mathbb{P}(\bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha) = 0$. Thus

$$\mathbb{P}\left(\bigcup_{\substack{\alpha>1 \\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha\right) = 0.$$

Let $\omega \in \Omega$ so that there exists $\alpha(\omega)$ for which $\liminf n^{\alpha(\omega)} U_n(\omega) < \infty$.Then take $1 < \tilde{\alpha}(\omega) < \alpha(\omega)$ with $\tilde{\alpha}(\omega) \in \mathbb{Q}$. We have that $\liminf n^{\tilde{\alpha}(\omega)} U_n(\omega) = 0$. Then for all

$n \in \mathbb{N}$ there exists $m(\omega) > n$ so that $m^{\tilde{\alpha}} U_m(\omega) < 1$. Thus, $\omega \in \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha}$. Finally we have that

$$\begin{aligned}\{(\exists \alpha > 1) \liminf n^{\alpha} U_n \in \mathbb{R}\} \subseteq \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha} \\ \Rightarrow \mathbb{P}\left((\exists \alpha > 1) \liminf n^{\alpha} U_n \in \mathbb{R}\right) = 0.\end{aligned}$$

ii. We will use Borel-Cantelli 2) (Skript Lemma 3.1 p. 36). Define $A_n = \{U_n \leq n^{-1}\}$, it's clear that $(A_n)_{n\in\mathbb{N}}$ are independent. We have $\mathbb{P}(A_n) = \frac{1}{n}$, then $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$. By Borel-Cantelli

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k\right) = 1 > 0.$$

Additionally, if $\omega \in \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$, for all $n \in \mathbb{N}$ there exists $k_n(\omega) > n$ so that $k_n(\omega) U_{k_n(\omega)} \leq 1$. Thus, $0 \leq \liminf n U_n \leq 1$. To conclude:

$$\begin{aligned}\bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha} \subseteq \{\liminf n U_n \in \mathbb{R}\} \\ \Rightarrow \mathbb{P}\left(\liminf n U_n \in \mathbb{R}\right) = 1 > 0.\end{aligned}$$

Q3. STRONG LAW OF LARGE NUMBER FOR VARIABLE WITH 4TH MOMENT. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space . Take $(X_n)_{n\in\mathbb{N}}$ a series of independent identically distributed random variables. Suppose that $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^4) < \infty$, and define $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) Prove that $\mathbb{E}(S_n^4) = \frac{1}{n^3}\mathbb{E}(X_1^4) + \frac{6(n-1)}{n^3}\mathbb{E}(X_1^2)^2$. Why $\mathbb{E}(X_1^2) < \infty$?.

(b) Show that

$$\mathbb{P}(|S_n| > a) \leq \frac{6}{a^4}\frac{1}{n^2}\mathbb{E}\left(X^4\right).$$

(c) Using Borel-Cantelli show that $\mathbb{P}(\lim S_n = 0) = 1$.

(d) Now if the hypothesis $\mathbb{E}(X_1) = 0$ is changed. Prove that $\lim S_n = \mathbb{E}(X_1)$.

Solution

(a) We have that

$$\mathbb{E}\left(S_n^4\right) = \frac{1}{n^4}\sum_{i,j,k,l=1}^{n} \mathbb{E}\left(X_i X_j X_k X_l\right),$$

note that if $i \notin \{j, k, l\}$

$$\mathbb{E}\left(X_i X_j X_k X_l\right) = \mathbb{E}\left(X_i\right)\mathbb{E}\left(X_j X_k X_l\right) = 0.$$

Thus,

$$\mathbb{E}\left(X_i X_j X_k X_l\right) = \frac{1}{n^4}\sum_{i,j,k,l=1}^{n} \mathbf{1}_{\{i=j=k=l\}}\mathbb{E}\left(X_1^4\right) + \frac{1}{n^4}\sum_{i,j,k,l=1}^{n} (\mathbf{1}_{\{i=j\neq k=l\}} + \mathbf{1}_{\{i=k\neq j=l\}} + \mathbf{1}_{\{i=l\neq k=j\}})\mathbb{E}\left(X_1^2\right.$$

$$= \frac{1}{n^3}\mathbb{E}\left(X_1^4\right) + \frac{6(n-1)}{n^3}\mathbb{E}\left(X_1^2\right)^2.$$

We have by Hölder inequality that

$$\mathbb{E}\left(X_1^2\right) = \mathbb{E}\left(X_1^2 * 1\right) \leq \sqrt{\mathbb{E}\left(X_1^4\right)}\sqrt{\mathbb{E}\left(1^2\right)} < \infty$$

(b) We will use the Markov inequality, i.e.,

$$\begin{aligned}
\mathbb{P}\left(|S_n| > a\right) &= \mathbb{P}\left(\frac{(S_n)^4}{a^4} > 1\right) \\
&= \mathbb{E}\left[\mathbf{1}_{\left\{\frac{(S_n)^4}{a^4}\geq 1\right\}}\right] \\
&\leq \mathbb{E}\left[\frac{(S_n)^4}{a^4}\mathbf{1}_{\left\{\frac{(S_n)^4}{a^4}\geq 1\right\}}\right] \\
&\leq \mathbb{E}\left[\frac{(S_n)^4}{a^4}\right] \\
&= \frac{1}{a^4}\left(\frac{1}{n^3}\mathbb{E}\left(X^4\right) + \frac{6(n-1)}{n^3}\mathbb{E}\left(X_1^2\right)^2\right) \\
&\leq \frac{6}{a^4}\frac{1}{n^2}\mathbb{E}\left[X_1^4\right],
\end{aligned}$$

where in the last inequality we have used that $\mathbb{E}\left[X_1^2\right]^2 \leq \mathbb{E}\left[X_1^4\right]$.

(c) Take $A_n^m = \{\omega : |S_n(\omega)| > \frac{1}{m}\}$. We have that

$$\sum_{n\in\mathbb{N}} \mathbb{P}(A_n^m) \leq \sum_{n\in\mathbb{N}} 6m^4\frac{1}{n^2}\mathbb{E}\left(X^4\right) < \infty.$$

By Borel-Cantelli $\mathbb{P}(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k) = 0$, so

$$\mathbb{P}\left(\bigcup_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) = 0$$

$$\mathbb{P}\left(\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{k\geq n} A_k^c\right) = 1.$$

If $\omega \in \bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{n\geq k} A_k^c$ then for all $m \in \mathbb{N}$ there exists $n(\omega)$ so that for all $k \geq n(\omega)$ $|S_n(\omega)| < \frac{1}{m}$. Thus, $\lim_{n\to\infty} |S_n(\omega)| = 0$. This implies that

$$\mathbb{P}\left(\lim S_n(\omega) = 0\right) = 1.$$

(d) Define $\tilde{X}_n = X_n - \mathbb{E}\left(X_n\right)$. We have that $\tilde{X}_n$ satisfies all the hypothesis for (c), then

$$\mathbb{P}\left(\lim \tilde{X}_n = 0\right) = 1$$

$$\Rightarrow \mathbb{P}\left(\lim_{n\to\infty} X_n = \mathbb{E}\left[X_n\right]\right) = 1.$$