R. Hiptmair
L. Scarabosio
C. Urzua Torres

Spring Term 2015

Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 11

Problem 11.1 An Impossible Interpolation Estimate (Core problem)

[NPDE, Thm. 5.3.37] gave us bounds for the $L^2(\Omega)$-norm and $H^1(\Omega)$-seminorm of the error of piecewise linear interpolation on a triangular mesh of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$. These bounds invariably contained the $H^2(\Omega)$-norm of the interpolated function. Now, somebody claims to have found an analogous interpolation estimate of the form

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \leq C h_{\mathcal{M}} \rho_{\mathcal{M}} |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{11.1.1}$$

with some constant $C > 0$.

(11.1a) Show that (11.1.1) implies

$$\|\mathsf{I}_1 u\|_{L^2(\Omega)} \leq C \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{11.1.2}$$

with a constant $C > 0$ whose dependence of $h_{\mathcal{M}}$ and $\rho_{\mathcal{M}}$ should be made explicit.

HINT: First study [NPDE, Rem. 5.3.44].

(11.1b) Argue why (11.1.2) cannot be true.

HINT: Remember [NPDE, Ex. 2.4.18], [NPDE, Cor. 2.4.24]. Note that we are in a 2D setting.

Problem 11.2 Projection onto Constants (Core problem)

In [NPDE, Section 5.3.1] we derived L^2- and H^1-estimates for the error of piecewise linear interpolation on a grid, see [NPDE, Eq. (5.3.14)] and [NPDE, Eq. (5.3.16)]. The key tool was the integral representation formula [NPDE, Eq. (5.3.9)]. In this problem we practice these techniques for an even simpler projection operator.

Given a grid $\mathcal{M} := \{]x_{j-1}, x_j[\colon 1 \leq j \leq M\}$ of $[a, b] \subset \mathbb{R}$ we define a projection onto the space $\mathcal{S}_0^{-1}(\mathcal{M})$ of piecewise constant discontinuous functions on $\mathcal{M}$ according to

$$\mathsf{I}_0 : \begin{cases} L^2(]a,b[) & \mapsto \quad \mathcal{S}_0^{-1}(\mathcal{M}) \\ u & \mapsto \quad \sum_{j=1}^{M} \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(\xi)\, \mathrm{d}\xi \cdot \chi_{]x_{j-1},x_j[}, \end{cases} \tag{11.2.1}$$

where $\chi_{]x_{j-1},x_j[}$ stands for the characteristic function of the interval $]x_{j-1}, x_j[$, that is

$$\chi_{]x_{j-1},x_j[}(x) = \begin{cases} 1 & \text{, if } x \in]x_{j-1}, x_j[, \\ 0 & \text{else.} \end{cases} \tag{11.2.2}$$

We abbreviate $K :=]x_{j-1}, x_j[$ for some $j = 1, \dots, M$.

Remark: The linear projection I_0 is an instance of an L^2-projection. Generically, given a (closed) subspace $V \subset L^2(\Omega)$, the associated L^2-projection operator $\mathsf{Q}_V : L^2(\Omega) \mapsto V$ is defined through as solution operator of the variational problem

$$\mathsf{Q}_V u \in V : \quad \langle \mathsf{Q}_V u, v \rangle_{L^2(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} \quad \forall v \in V \ .$$

(11.2a) Compute $\mathsf{I}_0 u$ on $[0,1]$ for $u(x) = x$ and an equidistant mesh with meshwidth $h := M^{-1}$. Sketch the function $\mathsf{I}_0 u$.

(11.2b) Derive the local integral representation formula for the projection error

$$(u - \mathsf{I}_0 u)(x) = \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \int_y^x u'(\xi) \, \mathrm{d}\xi \, \mathrm{d}y, \quad x_{j-1} < x < x_j. \tag{11.2.3}$$

HINT: Use the fundamental theorem of calculus [NPDE, Eq. (2.5.2)].

(11.2c) Starting from (11.2.3) deduce the estimate

$$\|u - \mathsf{I}_0 u\|^2_{L^2(]x_{j-1},x_j[)} \leq |x_j - x_{j-1}|^2 |u|^2_{H^1(]x_{j-1},x_j[)}. \tag{11.2.4}$$

HINT: Apply the Cauchy-Schwarz inequality for integrals [NPDE, Eq. (2.3.15)] twice.

(11.2d) Based on (11.2.4) derive the global projection error estimate

$$\|u - \mathsf{I}_0 u\|_{L^2(]a,b[)} \leq h_{\mathcal{M}} |u|_{H^1(]a,b[)}, \tag{11.2.5}$$

where $h_{\mathcal{M}}$ is the meshwidth of $\mathcal{M}$.

Problem 11.3 Convergence of Finite Element Solutions (Core problem)

A student is testing his implementation of a finite element method. On the square domain $\Omega = (0,1)^2$ he considers the 2nd-order elliptic boundary value problem

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega, \\ u &= \frac{1}{4}\left(1 - \|\boldsymbol{x}\|^2\right) && \text{on } \partial\Omega. \end{aligned} \tag{11.3.1}$$

He computes an approximate solutions u_N by means of a finite element Galerkin method using linear (piecewise first order polynomials) and quadratic (piecewise second order polynomials) finite elements, denoted by LFE and QFE respectively, on a sequence of triangular meshes $\mathcal{M}$.

The following table lists the measured $H^1(\Omega)$-seminorm of the discretization error as a function of the meshwidth h.

h	0.70	0.35	0.17	0.088	0.044	0.022	0.011
LFE	0.10	0.051	0.025	0.012	0.0064	0.0032	0.0008
QFE	$1.75 \cdot 10^{-16}$	$1.24 \cdot 10^{-15}$	$5.71 \cdot 10^{-15}$	$2.29 \cdot 10^{-14}$	$8.91 \cdot 10^{-14}$	$3.53 \cdot 10^{-13}$	$1.41 \cdot 10^{-12}$

(11.3a) Show that $u(\boldsymbol{x}) = \frac{1}{4}(1 - \|\boldsymbol{x}\|^2)$ is the exact solution of (11.3.1)

(11.3b) What kind of convergence (qualitative and quantitative) for linear Lagragian finite elements can be inferred from the error table?

(11.3c) Explain the striking difference between the behavior of the discretization error for linear and quadratic Lagrangian finite elements.

Problem 11.4 Localized Interpolation Error Estimates

There is a more refined way than that of [NPDE, Thm. 5.3.37] to state interpolation error estimates. It relies on the piecewise constant *meshwidth function*

$$\hbar(\boldsymbol{x}) = h_K \quad \text{if} \quad \boldsymbol{x} \in K, \tag{11.4.1}$$

where K is a cell of a triangular mesh $\mathcal{M}$ of a domain $\Omega \subset \mathbb{R}^2$.

Based on [NPDE, Lemma 5.3.34] derive the estimate

$$\left\| \hbar^{-2}(u - \mathsf{I}_1 u) \right\|_{L^2(\Omega)} \leq C |u|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega), \tag{11.4.2}$$

What is a concrete value for the constant C?

Problem 11.5 Shape Regularity and Angle Condition

When deriving interpolation error estimates for linear interpolation on triangular meshes in [NPDE, Section 5.3.2], it was convenient to introduce the concept of a shape regularity measure ρ_K for a triangle K, see [NPDE, Def. 5.3.36]. The intuition is that in 2D the shape regularity measure indicates the degree to which a triangle is distorted. This distortion was linked to the angles in [NPDE, Fig. 198], [NPDE, Fig. 199], and [NPDE, Fig. 200]. This link will be explored in this problem.

Bound the smallest angle of a triangle K by an expression involving only ρ_K.

HINT: From secondary school recall the formula $|K| = \frac{1}{2} ab \sin(\gamma)$, where γ is the angle enclosed by the sides with lengths a, b.

Published on May 6.

To be submitted on May 13.

References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations".SVN revision # 75873.

[NCSE] Lecture Slides for the course "Numerical Methods for CSE".

Last modified on May 6, 2015