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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 13

Problem 13.1 The One-Dimensional Wave Equation (Core problem)

[NPDE, Section 6.2.2] introduced the one-dimensional wave equation with constant coefficients as a simple model for wave propagation. There, in [NPDE, § 6.2.20], we examined the so-called Cauchy problem, for which the spatial domain is the whole real line. In this problem we return to a bounded spatial domain and impose non-homogeneous Dirichlet boundary conditions that depend on time. F

The 1D wave equation with constant coefficients reads

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - c\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0, \quad \text{on } (0,1)\times(0,T)\ . \tag{13.1.1}$$

The partial differential equation is supplemented with Dirichlet boundary conditions and zero initial conditions

$$u(0,t) = 0, \quad u(1,t) = \begin{cases} \sin t & 0 \leq t \leq \pi, \\ 0 & \text{otherwise}, \end{cases}$$
$$u(x,0) = 0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(x,0) = 0.$$

This initial-boundary value problem can be tackled numerically using the method of lines, see [NPDE, Section 6.2.3], which, intermittently, leads to the ODE

$$\mathbf{M}\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\vec{\boldsymbol{\mu}} + \mathbf{A}\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\varphi}}(t), \tag{13.1.2}$$

for the time-dependent coefficient vector $\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\mu}}(t)$ associated with a spatial Galerkin discretization.

In this task we focus on finite element Galerkin discretization with piecewise linear Lagrangian finite elements on equidistant meshes, see [NPDE, Section 1.5.2.2].

The non-homogeneous Dirichlet boundary condition at $x = 1$ can be taken into account through the use of a locally supported *offset function* as explained in [NPDE, Rem. 1.5.80], see also [NPDE, Section 3.6.5].

(13.1a) Compute the stiffness matrix $\mathbf{A}$ and the mass matrix $\mathbf{M}$ using the trapezoidal rule [NPDE, Eq. (1.5.72)] to evaluate the integrals.

HINT: The mass matrix will be diagonal, because the use of this particular quadrature formula effects "mass lumping" [NPDE, Rem. 6.2.45]. The stiffness matrix has already been computed in [NPDE, Section 1.5.2.2].

(13.1b) Find a piecewise linear, time-dependent, and locally supported offset function g, which we can used to incorporate the nonhomogenous Dirichlet boundary condition into the semi-discrete problem.

HINT: [NPDE, Rem. 1.5.80] with additional dependence on time.

(13.1c) What is the right-hand side of the ordinary differential equation (13.1.2) arising from the method of lines?

HINT: The right-hand side will involve the offset function found in subproblem (13.1b).

(13.1d) Write a MATLAB function

```
U = wave(N, K, T, c)
```

that solves (13.1.1) with the method just described, with `N` interior nodes in space, `K` timesteps and final time `T`, and returns the results in a $(N+2) \times (K+1)$-matrix `U`. Use leapfrog timestepping from [NPDE, § 6.2.43], in particular [NPDE, Eq. (6.2.44)] and do not forget the special initial step.

(13.1e) Run your code with $N = 100$, $K = 2000$, $T = 7$ and $c = 1$. Plot the solution continually (MATLAB command `drawnow`) while solving (use the MATLAB `pause` command to slow down or halt execution so that you can actually see the "movie").

HINT: For debugging purposes: the value of the solution at point $(50, 2000)$ should be 0.0761.

(13.1f) As in [NPDE, Exp. 6.2.46], plot the (discrete) elastic, kinetic and total energies as a function of time.

HINT: Formulas are given in [NPDE, Section 6.2.4]. In particular, [NPDE, Code 6.2.48] may be useful.

(13.1g) Describe the behavior of the solution computed in subproblem (13.1e) in qualitative terms related to "wave propagation".

Problem 13.2 Crank-Nicolson Timestepping (Core problem)

In this problem we conduct some analysis of a two-step timestepping scheme for the semi-discrete wave equation.

As an alternative to leapfrog timestepping, for the typical method of lines ODE for wave propagation problems, see [NPDE, Section 6.2.3],

$$\mathbf{M}\ddot{\vec{\boldsymbol{\mu}}} + \mathbf{A}\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\varphi}}(t),$$

one may use the *Crank-Nicolson method*, in analogy to [NPDE, Eq. (6.2.41)] written as

$$\begin{aligned}\mathbf{M}\frac{\vec{\boldsymbol{\mu}}^{(j+1)} - 2\vec{\boldsymbol{\mu}}^{(j)} + \vec{\boldsymbol{\mu}}^{(j-1)}}{\tau^2} &= \\ &- \mathbf{A}\left(\frac{1}{4}\vec{\boldsymbol{\mu}}^{(j+1)} + \frac{1}{2}\vec{\boldsymbol{\mu}}^{(j)} + \frac{1}{4}\vec{\boldsymbol{\mu}}^{(j-1)}\right) + \frac{1}{4}\vec{\boldsymbol{\varphi}}(t_{j+1}) + \frac{1}{2}\vec{\boldsymbol{\varphi}}(t_j) + \frac{1}{4}\vec{\boldsymbol{\varphi}}(t_{j-1}). \quad (13.2.1)\end{aligned}$$

As was done for leapfrog in [NPDE, § 6.2.43], it can also be formulated as a single-step method,

$$\frac{\vec{\mu}^{(j+1)} - \vec{\mu}^{(j)}}{\tau} = \frac{1}{2}\left(\vec{\mathbf{v}}^{(j+1)} + \vec{\mathbf{v}}^{(j)}\right), \tag{13.2.2}$$

$$\mathbf{M}\frac{\vec{\mathbf{v}}^{(j+1)} - \vec{\mathbf{v}}^{(j)}}{\tau} = -\mathbf{A}\frac{\vec{\mu}^{(j+1)} + \vec{\mu}^{(j)}}{2} + \frac{\vec{\varphi}(t_{j+1}) + \vec{\varphi}(t_j)}{2}. \tag{13.2.3}$$

(13.2a) Show that (13.2.1) is equivalent to (13.2.2)–(13.2.3), i.e., they both give the same recursion for $\vec{\mu}^{(j)}$.

(13.2b) Show that (13.2.2)–(13.2.3) conserves the discrete energy,

$$E_j = \frac{1}{2}(\vec{\mathbf{v}}^{(j)})^\top \mathbf{M}\vec{\mathbf{v}}^{(j)} + \frac{1}{2}(\vec{\mu}^{(j)})^\top \mathbf{A}\vec{\mu}^{(j)}.$$

HINT: Take the scalar product of (13.2.2) with $\frac{1}{2}\mathbf{A}\left(\vec{\mu}^{(j)} + \vec{\mu}^{(j+1)}\right)$, and of (13.2.3) with $\frac{1}{2}\left(\vec{\mathbf{v}}^{(j)} + \vec{\mathbf{v}}^{(j+1)}\right)$, and then add them together.

Problem 13.3 Wave Equation with Perfectly Matched Layers

As in [NPDE, § 6.2.20] we consider Cauchy problem for the 1D wave equation (on the unbounded domain $\Omega = \mathbb{R}$):

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) &= 0 && \text{on } \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) &= v_0(x), && \text{on } \Omega, \end{aligned} \tag{13.3.1}$$

where

$$c(x) = 1 \quad \text{for } x \geq 1,\ x \leq -1, \quad \text{and} \quad \operatorname{supp}(u_0), \operatorname{supp}(v_0) \subset]-1, 1[.$$

We are only interested in that part of the solutions of (13.3.1) that lies inside $[-1, 1]$.Nevertheless we can not simply restrict problem (13.3.1) to $[-1, 1]$, for instance by imposing Dirichlet boundary conditions, since reflected waves would completely supersede the solution of the Cauchy problem after a short time. Such reflections could be seen in subproblem (13.1e).

Instead we have to use *absorbing boundary conditions* that let waves pass undisturbed. One option are so-called *perfectly matched layers* (PML). This technique is based on introducing a artificial material in a zone outside the region of interest that absorbes waves. In our example these zones are $[-1 - L, -1]$ and $[1, 1 + L]$.

The variational formulation of the PML augmented 1D wave equations then reads: seek $u(t) \in H^1(]-1-L, 1+L[)$ and $v(t) \in L^2([-1-L, 1+L])$ such that for all $w \in H^1(]-1-L, 1+L[)$ and $q \in L^2(]-1-L, 1+L[)$

$$\begin{aligned} \int_{-1-L}^{1+L} \frac{\partial u}{\partial t} w \,\mathrm{d}x + \int_{-1-L}^{1+L} \sigma(x) u\, w \,\mathrm{d}x + \int_{-1-L}^{1+L} v \frac{\partial w}{\partial x} \,\mathrm{d}x &= \int_{-1-L}^{1+L} v_0\, w \,\mathrm{d}x, \\ \int_{-1-L}^{1+L} \frac{\partial v}{\partial t} q \,\mathrm{d}x + \int_{-1-L}^{1+L} \sigma(x) v\, q \,\mathrm{d}x - \int_{-1-L}^{1+L} c^2(x)\frac{\partial u}{\partial x} q \,\mathrm{d}x &= 0, \end{aligned} \tag{13.3.2}$$

with

$$\sigma = \begin{cases} 0 & -1 < x < 1 \\ \sigma_0 & x < -1, x > 1, \quad \sigma_0 > 0 \end{cases}. \tag{13.3.3}$$

(13.3a) State the standard variational formulation of (13.3.1), if homogeneous Neumann boundary conditions are imposed at $x = \pm 1$.

(13.3b) State the variational problem (13.3.2) for $L = 0$, which means that the absorbing layer is ignored.

(13.3c) Show that the two variational formulations obtained in subproblem (13.3a) and subproblem (13.3b) are equivalent, when one makes the substitution

$$v(x,t) = c(x)^2 \int_0^t \frac{\partial u}{\partial x}(x,\tau)\,\mathrm{d}\tau \quad \Leftrightarrow \quad \frac{\partial v}{\partial t} = c(x)^2 \frac{\partial u}{\partial x}\,. \tag{13.3.4}$$

HINT: Test the right equation in (13.3.4) with a function in $L^2(]-1,1[)$.

Now we tackle the full discretization of (13.3.2) and we pursue the method-of-lines policy. In particular we resort to a Galerkin finite element discretization in space based on a on a mesh $\mathcal{M}$ with $N+1$ equidistant nodes $x_0 := L-1 \leq \cdots \leq x_N := 1+L$. Then, in (13.3.2), we replace the space $H^1(]-1-L,1+L[)$ with the space $\mathcal{S}_1^0(\mathcal{M})$ of piecewise linear continuous functions, see [NPDE, § 1.5.61]. As trial and test space for $L^2(]-1-L,1+L[)$ we choose the space $\mathcal{S}_0^{-1}(\mathcal{M})$ of piecewise constant discontinuous functions on $\mathcal{M}$. For both spaces we opt for the canonical local supported nodal basis functions: "tent functions" according to [NPDE, Eq. (1.5.62)] for $\mathcal{S}_1^0(\mathcal{M})$, and the characteristic functions of mesh cells for $\mathcal{S}_0^{-1}(\mathcal{M})$.

The resulting ODE system is discretized via a special variant of leapfrog timestepping, see [NPDE, § 6.2.43]. In each timestep it leads to the following discrete variational equation:

$$\begin{aligned}
\int_{-1-L}^{L+1} \frac{u_N^{(k+1)} - u_N^{(k)}}{\Delta t}\, w_N \,\mathrm{d}x + \int_{-1-L}^{L+1} \sigma \frac{u_N^{(k+1)} + u_N^{(k)}}{2}\, w_N \,\mathrm{d}x + \int_{-1-L}^{L+1} v_N^{(k)}\, \frac{\partial w_N}{\partial x} \,\mathrm{d}x \\
= \int_{-1-L}^{L+1} v_0\, w_N \,\mathrm{d}x, \\
\int_{-1-L}^{L+1} \frac{v_N^{(k+1)} - v_N^{(k)}}{\Delta t}\, q_N \,\mathrm{d}x + \int_{-1-L}^{L+1} \sigma \frac{v_N^{(k+1)} + v_N^{(k)}}{2}\, q_N \,\mathrm{d}x - \int_{-1-L}^{L+1} c^2 \frac{\partial u_N^{(k+1)}}{\partial x}\, q_N \,\mathrm{d}x = 0,
\end{aligned}$$

with $u_N, w_N \in \mathcal{S}_1^0(\mathcal{M})$ and $v_N, q_N \in \mathcal{S}_0^{-1}(\mathcal{M})$. This scheme will underly the implementation requested in this problem.

(13.3d) Derive first the linear system of equations that has to be solved in each timestep. Describe it using suitable Galerkin matrices and give formulas for their entries.

HINT: The formulas are simple: recall [NPDE, Eq. (1.5.64)] and [NPDE, Eq. (1.5.69)] and make use of the simplifications offered by the equidistant mesh.

(13.3e) Implement the scheme in `main_pml.m` for the inital data given there.

(13.3f) As in subproblem (13.1f) , plot the (discrete) elastic, kinetic and total energies as a function of time.

HINT: Formulas are given in [NPDE, Section 6.2.4]. In particular, [NPDE, Code 6.2.48] may be useful.

(13.3g) Describe the behavior of the solution computed in subproblem (13.3f) in qualitative terms related to "wave propagation".

Problem 13.4 Helmholtz Equation

So far we have almost always faced initial boundary value problems for the linear wave equation, with the exception of the 1D Cauchy problem from [NPDE, § 6.2.20], where boundary conditions did not occur. In this problem we learn about a situation, where we can drop initial conditions: the *time-periodic* setting.

Now we consider the linear wave equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial^2 u(\boldsymbol{x},t)}{\partial t^2} - \Delta u(\boldsymbol{x},t) &= f(\boldsymbol{x},t) \quad && \text{in } \Omega \times \mathbb{R}, \\ u(\boldsymbol{x},t) &= 0, && \text{on } \partial\Omega \times \mathbb{R}, \end{aligned} \tag{13.4.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$, but for all times $t \in \mathbb{R}$.

We assume a time-periodic excitation

$$f(\boldsymbol{x},t) = \mathrm{Re}\{\hat{f}(\boldsymbol{x})\mathrm{e}^{\imath\omega t}\}, \tag{13.4.2}$$

with angular frequency $\omega > 0$ (Re designated the real part). The function $\hat{f} : \Omega \to \mathbb{C}$, $\hat{f} \in L^2(\Omega)$, is called the complex amplitude/phasor of f.

(13.4a) Show that

$$u(\boldsymbol{x},t) = \mathrm{Re}\{\hat{u}(\boldsymbol{x})\mathrm{e}^{\imath\omega t}\}, \tag{13.4.3}$$

solves the variational form of (13.4.1), if $\hat{u} \in H^1(\Omega)$ solves

$$\begin{aligned} -\omega^2 \hat{u}(\boldsymbol{x}) - \Delta\hat{u}((\boldsymbol{x}) &= \hat{f}(\boldsymbol{x}) \quad && \text{in } \Omega, \\ \hat{u}(\boldsymbol{x}) &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{13.4.4}$$

(13.4b) In class we learned that the hyperbolic evolution governed by (13.4.1) involves an incessant conversion of elastic energy and kinetic energy given by

$$E_{el}(t) = \frac{1}{2}\int_\Omega \|\mathbf{grad}\, u(\boldsymbol{x},t)\|^2 \,\mathrm{d}\boldsymbol{x}, \tag{13.4.5}$$

$$E_{kin}(t) = \frac{1}{2}\int_\Omega \left|\frac{\partial u(\boldsymbol{x},t)}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x} \tag{13.4.6}$$

Show directly, without appealing to [NPDE, Thm. 6.2.29], but using (13.4.5) and (13.4.6), that for $u = u(\boldsymbol{x},t)$ according to (13.4.3) and (13.4.4) the total energy is preserved.

(13.4c) Give a formula for the mean elastic and kinetic energy of u given by (13.4.3), that is, express

$$\hat{E}_{el} := \frac{1}{T}\int_0^T \frac{1}{2}\int_\Omega \|\mathbf{grad}\, u(\boldsymbol{x},t)\|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{13.4.7}$$

$$\hat{E}_{kin} := \frac{1}{T}\int_0^T \frac{1}{2}\int_\Omega \left|\frac{\partial u(\boldsymbol{x},t)}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{13.4.8}$$

in terms of suitable expressions for $\hat{u}$, where $T = \frac{2\pi}{\omega}$ is the duration of one period.

(13.4d) For $\Omega = \{x \in \mathbb{R}^2 : \|x\| < 1\}$ and

$$\hat{f}(\boldsymbol{x}) = \begin{cases} \cos^2\left(2\pi\left\|\boldsymbol{x} - \binom{0.5}{0.5}\right\|\right) & \text{, if } \left\|\boldsymbol{x} - \binom{0.5}{0.5}\right\| < \frac{1}{4}, \\ 0 & \text{elsewhere.} \end{cases}$$

Implement a C++ code that solves (13.4.4) for $\omega = 10$ using piecewise linear Lagrangian FE.

HINT: As usual, you are required to use your previous implementations of `DofHandler`, `MatrixAssembler`, `VectorAssembler`, `BoundaryDofs`, and local assemblers, developed in subproblems 7.4, 8.1, and 8.2 (also available in the corresponding solution folders).

Now we reduce the homogeneous boundary condition in (13.4.1) with a special boundary condition, the so-called first order absorbing boundary condition

$$\mathbf{grad}\, u(\boldsymbol{x}, t) \cdot \mathbf{n}(\boldsymbol{x}) + \frac{\partial u}{\partial t}(\boldsymbol{x}, t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}. \tag{13.4.9}$$

(13.4e) Find a boundary condition for $\hat{u}$ in (13.4.4) such that, again, $u = u(\boldsymbol{x}, t)$ given by (13.4.3) solves the wave equation from (13.4.1) and satisfies (13.4.9).

(13.4f) Modify your C++ code so that it can handle the boundary conditions from subproblem (13.4e). Repeat the numerical experiment from subproblem (13.4d).

HINT: Observe here we are using complex numbers. This means you should be careful when declaring the triplets type and slightly modify your implementation of `LocalMass`. You may also use the templated class available in the lecture `svn` repository

`assignments_codes/assignment13/Problem4`

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References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations".SVN revision # 76119.

[NCSE] Lecture Slides for the course "Numerical Methods for CSE".

[LehrFEM] LehrFEM manual.

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