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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 5

Problem 5.1 Linear output functionals

In [NPDE, Section 2.2.3] we have seen that continuity ($\to$ [NPDE, Def. 2.2.49]) of linear forms with respect to energy norms ($\to$ [NPDE, Def. 2.2.38]) induced by symmetric positive definite bilinear forms ($\to$ [NPDE, Def. 2.2.35]) is a key property. Thus, for elliptic boundary value problems, continuity of linear forms in Sobolev spaces is crucial.

For the point evaluation functional, we investigated its continuity in $H^1(\Omega)$ in [NPDE, Ex. 2.4.18], for the source functional $v \to \int_\Omega f v \,\mathrm{d}\boldsymbol{x}$ continuity was studied in [NPDE, Section 2.3.3], whereas boundary functionals arising from non-homogeneous Neumann problems were examined in [NPDE, § 2.10.7].

In this problem we consider the linear functionals

$$J_1(v) := \int_\Omega \mathbf{c} \cdot \mathbf{grad}\, v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} , \quad \mathbf{c} \in \mathbb{R}^2 , \tag{5.1.1}$$

$$J_2(v) := \int_\Omega v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} , \tag{5.1.2}$$

$$J_3(v) := \int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x}) , \tag{5.1.3}$$

$$J_4(v) := \int_\Omega v\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) \mathrm{d}\boldsymbol{x} . \tag{5.1.4}$$

on the unit disk $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\}$. These functionals all make sense when we apply them to smooth functions.

Please answer the following questions for (5.1.1)–(5.1.4).

(5.1a) Which of these functionals are continuous on $L^2(\Omega)$? If you suspect a functional to be continuous, try to prove it. If you think, it is not continuous, try to find a counterexample as in [NPDE, § 2.4.20].

HINT: The functional (5.1.4) can be rewritten in terms of an integral over $\partial\Omega$.

(5.1b) Solve subproblem (5.1a), now with $L^2(\Omega)$ replaced with the Sobolev space $H^1(\Omega)$.

HINT: The standard tools for proving continuity of linear functionals on Sobolev spaces are the Cauchy-Schwarz inequality [NPDE, Eq. (2.2.39)] and trace theorems like [NPDE, Thm. 2.10.8].

Problem 5.2 Heat Conduction with Non-Local Boundary Conditions

This problem is meant to practice the conversion of a variational problem into a boundary value for a partial differential equation, see [NPDE, Section 2.5] and the extraction of boundary conditions hidden in the variational formulation as in [NPDE, Ex. 2.5.18].

Concretely, we consider the modelling of a two-dimensional cross-section of a submerged insulated wire, see Figure 5.1. The wire has a central core of conducting material, say copper, which carries a current. Ohmic losses lead to heat generation in the copper. Copper conducts heat very well and, thus, the copper core can be assumed to have a *uniform but unknown* temperature.

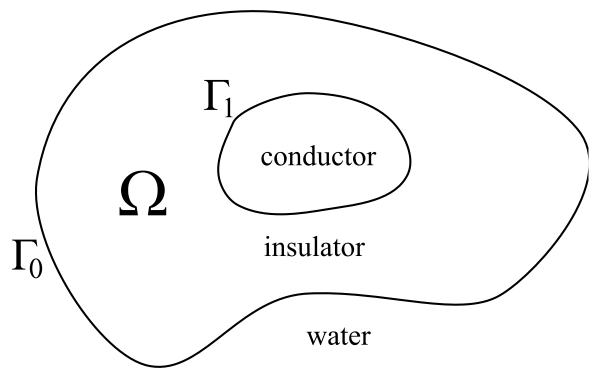

Figure 5.1: Cross-section of a submerged wire.

The copper is surrounded by an annulus of insulator, some plastic, for example, which is again surrounded by water, which we assume to be at a constant temperature of 0. We seek a mathematical model providing us with the temperature distribution within the insulation. Such a model is given by the variational problem

$$u \in V_0 : \int_\Omega \kappa(\mathbf{x}) \,\mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\Gamma_1} v(\mathbf{x}) \,\mathrm{d}S\,, \quad \forall v \in V_0, \tag{5.2.1}$$

where the heat conductivity κ is uniformly positive ($\rightarrow$ [NPDE, Def. 2.2.15]) and bounded, and with

$$V_0 = \left\{ v \in H^1(\Omega) \,\middle|\, v|_{\Gamma_0} = 0,\, v|_{\Gamma_1} = \text{const} \right\}.$$

(5.2a) Determine a bilinear form a and linear form ℓ so that (5.2.1) becomes an abstract linear variational problem $\mathsf{a}(u, v) = \ell(v)$.

(5.2b) Show that ℓ is continuous with respect to the energy norm induced by a, *cf.* [NPDE, Def. 2.2.49]. In the lecture we found this to be an essential condition for the well-posedness of a linear variational problem, see [NPDE, Lemma 2.2.47].

HINT: The energy norm is defined as in [NPDE, Def. 2.2.38], and ℓ must satisfy [NPDE, Eq. (2.2.48)] to be continuous with respect to this norm. Then use the *trace theorem* [NPDE, Thm. 2.10.8].

(5.2c) If u solves (5.2.1) and is sufficiently smooth, it also satisfies a partial differential equation on Ω. Find this equation.

HINT: Follow the approach of [NPDE, Section 2.5]: as test functions v use functions in $C_0^1(\Omega)$, that is, they should be zero on both boundaries Γ_0, Γ_1. Use [NPDE, Thm. 2.5.9] (with $\mathbf{grad}\, u$ in place of $\mathbf{j}$). Argue what happens to the boundary terms. Then appeal to [NPDE, Lemma 2.5.12].

(5.2d) The function u from problem (5.2c) must also satisfy a certain non-local boundary condition implied by (5.2.1). Find this boundary condition.

HINT: Follow the strategy from [NPDE, Ex. 2.5.18] and use the PDE derived in the previous sub-problem.

(5.2e) What is the physical interpretation of the boundary condition from (5.2d) in terms of heat conduction?

Problem 5.3 Minimization of a Quadratic Functional

[NPDE, Section 2.2.3] introduced abstract quadratic minimization problems, see [NPDE, Def. 2.2.24] and [NPDE, Def. 2.2.29]. As concrete examples arising from equilibrium models we studied quadratic minimization problems posed on the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ of scalar functions. In [NPDE, Section 2.4], we learned how to convert a quadratic minimization problem into variational form, see [NPDE, Eq. (2.4.9)]. [NPDE, Section 2.5] taught us how to use multidimensional integration by parts [NPDE, Thm. 2.5.9] to convert the linear variational problems on Sobolev spaces into a boundary value problems for 2$^{\text{nd}}$-order elliptic PDEs. In this exercise we practise all these steps in the case of an "exotic" quadratic minimization problem.

We consider the quadratic functional

$$J(\mathbf{u}) = \int_\Omega \left|\operatorname{div}\mathbf{u}(\mathbf{x})\right|^2 + \|\mathbf{u}(\mathbf{x})\|^2 + \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \tag{5.3.1}$$

with $\Omega \subset \mathbb{R}^3$ bounded, and for functions $\mathbf{u} : \Omega \to \mathbb{R}^3$, that is, J takes *vector field arguments*.

(5.3a) Identify the bilinear form a and the linear form ℓ in the quadratic functional J, *cf.* [NPDE, Def. 2.2.24].

HINT: See [NPDE, Def. 2.2.24].

(5.3b) Show that the bilinear form a from subproblem (5.3a) is symmetric and positive definite, see [NPDE, Def. 2.2.35].

HINT: See [NPDE, Eq. (2.2.26)] and [NPDE, Def. 2.2.35].

(5.3c) Show that the linear form ℓ from subproblem (5.3a) is continuous with respect to the energy norm induced by a.

HINT: The energy norm is defined as in [NPDE, Def. 2.2.38], and ℓ must satisfy [NPDE, Eq. (2.2.48)] to be continuous with respect to this norm.

(5.3d) Explain why the Sobolev space

$$H(\operatorname{div},\Omega) := \left\{\mathbf{v} : \Omega \to \mathbb{R}^3 \text{ integrable} \,\middle|\, \int_\Omega \left|\operatorname{div}\,\mathbf{v}\right|^2 + \|\mathbf{v}\|^2\,\mathrm{d}\mathbf{x} < \infty\right\}.$$

provides the right framework for studying the minimization problem for the functional J from (5.3.1).

(5.3e) Derive and state the linear variational problem equivalent to the minimization problem

$$\mathbf{u}_* = \underset{\mathbf{v}\in H(\operatorname{div},\Omega)}{\operatorname{argmin}} J(\mathbf{v}).$$

HINT: See [NPDE, Eq. (2.4.8)] and [NPDE, Eq. (2.4.9)].

(5.3f) Derive the *partial differential equation* on Ω that arises from the variational problem from (5.3e).

HINT: Follow the approach of [NPDE, Section 2.5], in particular [NPDE, Ex. 2.5.18]: as test functions $\mathbf{v}$ use vector fields in $(\mathcal{C}_0^1(\Omega))^3$, that is, they should be zero on the boundary. Use [NPDE, Thm. 2.5.9] (with $\operatorname{div} \mathbf{u}$ in place of v and $\mathbf{v}$ in place of $\mathbf{j}$) in order to "shift the div from $\mathbf{v}$ onto $\operatorname{div} u$ as $-\operatorname{grad}$". Argue, what happens to the boundary terms. Then appeal to [NPDE, Lemma 2.5.12].

(5.3g) The variational problem from (5.3e) also implies boundary conditions. Which?

HINT: Follow the strategy from [NPDE, Ex. 2.5.18] and use the PDE derived in subproblem (5.3f).

Problem 5.4 Poisson equation in polar coordinates

In the problem we will come across an important case of *transformation* of the domain of a boundary value problem prior to its discretization. We interpret the domain transformation as a *change of coordinates*, studying the concrete case of *polar coordinates* to switch from the unit disk domain to a simple square domain.

Remark. A rationale for using polar coordinates when dealing with boundary value problems on a disk is that, of course, mesh generation is trivial for a square and boundary approximation is not a concern. This will become in [NPDE, Chapter 3].

As a model problem we consider homogeneous Dirichlet problem for the Poisson equation [NPDE, Eq. (2.5.15)]

$$-\Delta u = f \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{5.4.1}$$

on the unit disk

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < 1\}\,. \tag{5.4.2}$$

Here Δ is the Laplace operator, see [NPDE, Rem. 2.5.14]. The variational (weak) formulation of (5.4.1) has been discussed in [NPDE, Ex. 2.9.2].

The transformation from polar coordinates (r, ϕ), $0 \le r \le 1$, $0 \le \phi < 2\pi$, to Cartesian coordinates $(x_1, x_2) \in \mathbb{R}^2$ is given by the mapping

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boldsymbol{\Phi}(r, \phi) := r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{5.4.3}$$

(*cf.* [NPDE, Eq. (2.4.21)]), and we have $\Omega = \boldsymbol{\Phi}(\Omega_p)$, with the tensor product domain

$$\Omega_p := [0,1] \times [0, 2\pi]. \tag{5.4.4}$$

Before you start solving this problem, we suggest you to refresh your knowledge about the polar coordinate example in [NPDE, § 2.4.20].

(5.4a) For a function $u \in C^1(\bar{\Omega})$. Compute the *Cartesian* components of $\mathbf{grad}\, u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^T$, for $u = u(r, \phi)$ given in *polar* coordinates, in terms of the partial derivatives $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \phi}$.

HINT: Use the chain rule for differentiation.

(5.4b) Explain the origin of the r-factor in the integration formula in polar coordinates:

$$\int_\Omega u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_0^1 \int_0^{2\pi} u(r,\phi) r \,\mathrm{d}\phi\,\mathrm{d}r. \tag{5.4.5}$$

HINT: You may appeal to the transformation formula for multi-dimensional integrals that you learned in your Analysis course.

(5.4c) As we learned in [NPDE, Section 2.9], the bilinear form associated with the homogeneous Dirichlet problem for the linear scalar 2nd-order differential operator $-\Delta$ on Ω reads:

$$\mathsf{a}(u,v) = \int_\Omega \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\;, \quad u,v \in H_0^1(\Omega)\;. \tag{5.4.6}$$

Rewrite it in terms of polar coordinates, that is, for $u = u(r,\phi)$ and $v = v(r,\phi)$, in terms of partial derivatives with respect to r and ϕ, and by means of an integral over the domain Ω_p as given in (5.4.4).

(5.4d) Let Ω_p be as in (5.4.4). Assuming that $u_p \in C^1(\bar{\Omega}_p)$, what further condition does u_p have to satisfy in order to ensure that $|u|_{H^1(\Omega)} < \infty$, where $u(x_1,x_2) := u_p(r(x_1,x_2),\phi(x_1,x_2)) : \Omega \to \mathbb{R}$ (and (r,ϕ) are the polar coordinates on Ω as given in (5.4.3))?

HINT: Use the results from subproblem (5.4c).

Write $u \in H_0^1(\Omega)$ for the weak solution on (5.4.1), and $u_p : \Omega_p \to \mathbb{R}$ for its transformation into polar coordinates: $u_p(r,\phi) := u(x_1(r,\phi), x_2(r,\phi))$.

(5.4e) What linear variational problem on Ω_p is solved by u_p? Assume that also f is given in polar coordinates: $f = f(r,\phi)$.

HINT: The results from task (5.4c) may come handy.

Now we assume that the source function enjoys rotational symmetry, i.e. $f = f(r)$, with no dependence on ϕ. Then the solution to (5.4.1) will also be rotationally symmetric: $u_p = u_p(r)$, $0 \le r \le 1$.

(5.4f) What variational problem (in polar coordinates) has to be satisfied by the rotationally symmetric solution $u_p = u_p(r)$ of (5.4.1) in the case of $f = f(r)$?

(5.4g) The *energy space* for the variational problem from task (5.4f) is:

$$V := \left\{ v \in L^2(]0,1[) : \int_0^1 r\left|\frac{dv}{dr}(r)\right|^2 \mathrm{d}r < \infty,\; v(1) = 0 \right\}. \tag{5.4.7}$$

Is the linear functional $J : V \to \mathbb{R}$ given by the point evaluation $J(v) = v(0)$ continuous on V?

HINT: Follow the approach of [NPDE, § 2.4.20] and try to find a function $v \in V$ with "$v(0) = \infty$". It is worth studying [NPDE, § 2.4.20] carefully, because after transformation back to the disk Ω, V can be regarded as the space of rotationally symmetric functions in $H_0^1(\Omega)$.

(5.4h) Assuming that $u_p \in \mathcal{C}^2([0,1])$, state the 2-point boundary value problem associated to the variational formulation from task (5.4f).

HINT: The boundary conditions will look strange, but, in light of the discussion in [NPDE, Rem. 2.3.6], the result of subproblem (5.4g) should make clear, why imposing boundary values at 0 does not make sense.

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References

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